

Combinatorics
MAD4203 3214

Class-C

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Welcome. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

Use $\mathcal{S}(N, K)$ for 2Stirling #s, and use $\mathcal{c}(N, K)$ for the signless-1Stirling #s.

C1: Short answer. Show no work.

a A *minimum* requirement for an LOR (letter-of-recommendation) from Prof. King is two courses. Circle:
 Yes True Darn tootin'!

b Written with \sum notation, the number of derangements of $[1..17]$ is: $\sum_{k=0}^{17} \binom{17}{k} \cdot [17-k]! \cdot [-1]^k$,
 which equals $[17]! \cdot \sum_{k=0}^{17} \frac{[-1]^k}{k!}$.

c Let \mathbb{D}_N be the set of derangements in \mathbb{S}_N . Then $|\mathbb{D}_4| = 24 - 24 + 12 - 4 + 1 = 9 = \begin{bmatrix} 2143 \\ 2341 \\ 2413 \end{bmatrix} \times 3$.
 The set of *good* k , st. \mathbb{D}_k has both odd perms *and* even perms, is $[4.. \infty)$.

Soln: Evidently 1,2,3 are bad [the only 3-derangement is a 3-cycle, even].
 Each $n \geq 4$ is good, since signatures $[n^1]$ and $[2^1, (n-2)^1]$ have opposite parities. And each is the sig of a derangement, since $n \geq 4$.

d There are $3 \binom{N}{2} = 3 \frac{N^2-N}{2}$ digraphs on vertex-set $[1..N]$.
 The number of *tournaments* (complete digraphs) on $[1..N]$ is $2 \binom{N}{2} = 2 \frac{N^2-N}{2}$.

e For posint N , let G_N be K_N (complete graph) but with one edge removed (so G_N has N vertices). Its chromatic polynomial is $\mathcal{P}_{G_N}(x) =$ _____
 [You may use rising/falling factorial in your answer, if you wish.]
 In particular, $\mathcal{P}_{G_4}(x) =$ _____

Chromatic: With v, w the endpts of the removed edge, the colorings of G_N split into those with: *Different colors for v, w , i.e. coloring K_N . Same colors for v, w , i.e. coloring K_{N-1} (by identifying duplicate-edges).* Hence

$$\begin{aligned} \mathcal{P}_{G_N}(x) &= \mathcal{P}_{K_N}(x) + \mathcal{P}_{K_{N-1}}(x) \\ &= [x \downarrow N] + [x \downarrow N-1] \\ &= [x - [N-1]] + 1 \cdot [x \downarrow N-1] \\ &= [x - [N-2]] \cdot [x \downarrow N-1]. \end{aligned}$$

In particular, $\mathcal{P}_{G_4}(x) = x[x-1][x-2]^2$; or just a attach a new vertex to a triangle.

C2: OYOP: In grammatical English *sentences*, write your essay on every *third* line (usually), so that I can easily write between the lines. Please number the pages "1 of 57", "2 of 57"... (or "1/57", "2/57"...). I suggest you put your name on each sheet.

i Prove that every tournament, G , on $[1..N]$ admits a **Hamiltonian** path; a path that visits each vertex exactly once (so the path will have $N-1$ edges).

ii Prove that every *DiG-connected* N -tournament, G , admits a **Hamiltonian cycle**; the cycle will have N edges.

End of Class-C

C1: _____ 155pts
C2: _____ 70pts
Total: _____ 225pts

HONOR CODE: "I have neither requested nor received help on this exam other than from my professor (or his colleague)."
 Name/Signature/Ord

Ord: _____