The Boole Transformation

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/

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ABSTRACT: The Boole map preserves Lebesgue measure on the one-point compactification $\dot{\mathbb{R}}$ of $\mathbb{R}$. The scaled Boole map preserves the finite measure having density $1/[x^2+1]$.

A condition for an endomorphism of $\mathbb{R}$ to be measure-preserving. Fix an open subset $X \subset \mathbb{R}$. When does a continuously differentiable (not necessarily invertible) map $f{:}X\circlearrowleft$ preserve Lebesgue measure? (N.B. For simplicity, we assume henceforth that f is everywhere finite-to-one.) Evidently f is measure-preserving exactly when

$$\forall x \in X: \qquad \sum_{r \in f^{-1}(x)} \frac{1}{|f'(r)|} \;=\; 1\,.$$

For consider a small interval I centered at x. The inverse images of x, write them $r_1 < \ldots < r_N$, have some minimum distance between them and so, if I is small enough, then $f^{-1}(I)$ consists of N disjoint intervals whose lengths are essentially $\left\{\mathrm{Len}(I)/f'(r_n)\right\}_{n=1}^{N}$. The intervals have "exactly" these lengths when I is infinitesimal. This justifies the equation and shows incidentally that f' must be everywhere non-zero.

More generally, given a cts density function $\Delta{:}X\to\mathbb{R}_+$, when does f preserve the measure determined by $\Delta(\cdot)$? (i.e, the measure which assigns to each set $B \subset X$ the value $\int_B \Delta(x)\cdot \mathrm{d}x$.) Well, first assume that on X the derivative of our function never vanishes. The condition equivalent to f preserving the measure from Δ is this: $\forall x \in X$,

$$\textbf{1:} \qquad \sum_{r:\, f(r)=x} \frac{\Delta(r)}{|f'(r)|} \;=\; \Delta(x)\,.$$

(Recall that $\Delta(\cdot)$ is continuous.)

Defn. Let $\dot{\mathbb{R}}$ be the one-point compactification of the reals. Define rational maps $f, h{:}\dot{\mathbb{R}}\to\dot{\mathbb{R}}$ by

$$f \;:=\; \left[x \mapsto x - \frac{1}{x}\right]; \qquad \text{(Boole map)}$$
$$h \;:=\; \tfrac{1}{2}\cdot f\,. \qquad \text{(Scaled Boole map)} \;\square$$

2: Theorem. *The Boole map preserves Lebesgue measure.* ♢

Proof. Lebesgue measure has density $\Delta(\cdot) \equiv 1$.

Fix a real x. Then $f(r) = x$ implies that $\frac{[r^2-1]}{r} = x$ and thus

$$\ddagger\textbf{:} \quad F_x(r) = 0, \quad \text{where} \quad F_x(r) := r^2 - xr - 1\,.$$

The discriminant of F_x is $[\text{-}x]^2 - 4\cdot 1\cdot[\text{-}1]$. Now x^2+4 is always positive, so F_x has two distinct real roots; call them ℓ and u. We need to verify that the sum in (1) is always 1 i.e, that

$$*\textbf{:} \quad \frac{1}{f'(\ell)} + \frac{1}{f'(u)} - 1$$

is <u>zero</u>. But $(*)$ equals

$$\frac{\ell^2}{\ell^2+1} + \frac{u^2}{u^2+1} - 1 \;=\; \frac{\ell^2}{\ell^2+1} - \frac{1}{u^2+1} \;=\; \frac{[\ell\cdot u]^2 - 1}{\text{Common denom}}\,.$$

Since the product of the two roots of a quadratic equals the constant term of the quadratic, here $\ell\cdot u$ equals -1 (from defn (‡)) and so the above numerator $[\ell\cdot u]^2 - 1$ is zero, as desired. ♦

3: Theorem. *The scaled Boole map preserves the measure having density*

$$\Delta(x) \;:=\; \frac{1}{x^2+1} \;\overset{\text{note}}{=\!=}\; \arctan'(x)\,.$$

(One could make this a probability density by dividing it by π so as to give total measure 1 to $\mathbb{R}$.) ♢

Proof. Fixing $x \in \mathbb{R}$, the eqn $x = h(r)$ means $2x = f(r)$ and thus has two distinct real solutions, $r := \ell$ and $r := u$.

From $x = h(r) = [r^2 - 1]/2r$ we compute

$$\begin{aligned} x^2 + 1 &= \frac{[r^2-1]^2}{4r^2} + 1 = \frac{[r^2-1]^2 + 4r^2}{4r^2} \\ &= \frac{[r^2+1]^2}{4r^2}\,. \end{aligned}$$

Taking reciprocals, $\Delta(x)/2 = 2r^2/[r^2+1]^2$ which equals $\Delta(r)/h'(r)$. But this equality holds when r is either of the two inverse images, ℓ and u, of x. Thus

$$\begin{aligned} \Delta(x) &= \frac{\Delta(x)}{2} + \frac{\Delta(x)}{2} \\ &= \frac{\Delta(\ell)}{h'(\ell)} + \frac{\Delta(u)}{h'(u)}\,. \end{aligned}$$

And this is precisely what (1) requires for h to preserve the from-Δ measure. ♦

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