Bipartite Matching

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ABSTRACT: Here is my proof (probably the standard proof) of a "Marriage Lemma" type condition which guarantees a matching in a bipartite graph. As a corollary, when applied to a finite bipartite graph one obtains the result that a doubly-stochastic matrix is a convex average of permutation matrices.

It also has two proofs of the "Injection Theorem", which might be an appropriate Putnam problem.

Prolegomenon. In a graph[♡1] $\Gamma = (\mathbf{V}, \mathsf{E})$, use E both for the edge *set* as well as the edge *relation*. I.e $v \mathsf{E} w$ means that vertices v and w are connected by an edge. Suppose the vertex set is. Our Γ is a ***bipartite graph*** if $\mathbf{V}$ can be written as a disjoint union $\mathbf{V} = \mathbb{B} \sqcup \mathbb{G}$ st.:

Whenever $x\mathsf{E}y$*, then* $[x \in \mathbb{B}$ *IFF* $y \in \mathbb{G}]$.

OUR GOAL is to place a condition on E which guarantees the existence of an involution $f{:}\mathbf{V}\circlearrowleft$ satisfying

$$\forall v \in \mathbf{V}: \quad v \,\mathsf{E}\, f(v)\,.$$

In this case, we say that the graph **"**admits a marriage**"**.

We first need a condition for graphs without loops. Graph $\Gamma = (\mathbf{V}, \mathsf{E})$ is a ***good tree*** if it is a tree (a connected acyclic graph) where each vertex has only countable degree. Moreover, Γ has no vertex of degree-zero (i.e, is not a single point) and has at most one ***root***, a vertex of degree-one. Say that Γ is a ***good forest*** if its connected components are good trees.

1: Forest Lemma. *Each good forest* $\Gamma = (\mathbf{V}, \mathsf{E})$ *admits a marriage.* ◊

Proof. Since a vertex has countable degree, each connected component in the forest has but countably many vertices and edges. So we may assume[♡2] that our forest is countable. (or is just one tree) if desired. Since a tree has no loops, the limitation on roots implies that each tree is infinite.

Enumerate $\mathbf{V}$ as $v_1, v_2, \ldots$. Iteratively do the following:

Let v *denote the lowest unmatched vertex* (*initially* v_1) *and let* C *denote the connected component of* E *owning* v.

a: If C has no root then choose some w with $v\mathsf{E}w$. Define f to match v with w and delete both vertices and all their edges.

b: If C has a root, x, then have f match x with w, where w denotes the unique vertex connected to x. Delete both vertices and their edges.

Whether deletion (a) or (b) is done, the resulting graph remains a good forest. Moreover, step (b) decreases by one the distance from the root in the component owning v to vertex v. So in but finitely many applications of (b), vertex v will be matched.

Thus f is eventually defined on all vertices, as desired. ♦

2: *Defn.* A ***weighted graph*** $\Gamma = (\mathbf{V}, \mathsf{E}; \mathcal{W})$ has a ***weight function*** $\mathcal{W}{:}\mathsf{E}{\to}[0,1]$ so that for each vertex v:

$$\text{2a:} \qquad \sum_{\mathsf{e}:\, \mathsf{e} \in \mathsf{E}\langle v\rangle} \mathcal{W}(\mathsf{e}) \;=\; 1$$

where $\mathsf{E}\langle v\rangle$ denotes the set of edges incident on v.

A ***loop*** in Γ is a cycle of distinct vertices and edges,

$$\text{2b:} \qquad v_0\, \mathsf{e}_0\, v_1\, \mathsf{e}_1\, v_2\, \mathsf{e}_2\, \ldots\, v_{N-1}\, \mathsf{e}_{N-1}$$

where subscripts are taken modulo N (i.e, $v_N = v_0$) and where e_i connects v_i with v_{i+1}. Moreover, each edge has *positive* weight. □

If we delete all zero-weight edges, then each vertex has countable degree. Hence (see preceding Axiom of Choice footnote) we may assume that each connected component is countable.

3: Main Theorem. *A weighted bipartite*[♡3] *graph* $\Gamma = (\mathbf{V}, \mathsf{E}; \mathcal{W})$ *admits a marriage.* ◊

[♡1] Here, a graph has no multiple-edges and no edges from a vertex to itself. We allow both $\mathbf{V}$ and E to be infinite.

[♡2] Stan Wagon points out that we appear to use the Axiom of Choice to pick out one vertex from each component from which to enumerate edges and conclude that the component is countable. In his book he shows that a bipartite theorem [implied by the one below] is equivalent to what he calls "The Axiom of Choice for pairs".

[♡3] There are weighted graphs which are not bipartite, e.g, 3 vertices, each connected to his fellows with a $\frac{1}{2}$-weight edge. Of course, a non-bipartite graph permits no marriage.

We may assume that Γ is connected. So deleting the zero-weight edges, henceforth $\boxed{\Gamma \text{ is countable}}$.

Given fncs $\mathcal{W},\mathcal{U}:\mathsf{E}\to[0,1]$, write[♡4] $\mathcal{W} \gg \mathcal{U}$ if for each vertex v: $[\mathcal{W}(v)=0]$ *implies* $[\mathcal{U}(v)=0]$. In particular, every $\mathcal{U}$-loop is a $\mathcal{W}$-loop.

3a: Sand-shifting lemma. *Consider L, a loop (2b) in the above Γ. Then there exists a weight-fnc $\mathcal{U} \ll \mathcal{W}$ for which L is no longer loop. (I.e, $\mathcal{U}$ gives zero-weight to some L-edge.)* ♢

Proof. Because Γ is bipartite, the loop length N must be even. Let μ be the minimum, around the loop, of the positive numbers

3b: $\quad 1-\mathcal{W}(\mathsf{e}_i)$, for i even; $\quad \mathcal{W}(\mathsf{e}_i)$, for i odd.

Alter the weights around the loop by adding μ to the weight of even edges and subtracting it from the weight of odd edges. Now, courtesy (2a), some edge e_i in the loop has weight 0 or 1. In the latter case, the edges e_{i-1} and e_{i+1} now have weight-zero. Thus deleting all edges of weight-zero eliminates this loop. ♦

3c: Edge-deletion lemma. *Fix an edge c in the above Γ. Then there exists a weight-fnc $\mathcal{U}\ll\mathcal{W}$ with $\mathcal{U}(\mathsf{c})=0$.* ♢

Proof. Say a (2b)-loop "contains" c if $\mathsf{e}_0=\mathsf{c}$. Enumerate the loops which contain c; there are but countably many since each loop is determined by a finite subset of the edge set. For each loop in turn, apply the above "weight shifting" procedure; after the k^{th} loop has been broken, let $\mathcal{W}_k(\cdot)$ denote the resulting new weight-fnc. So

$$\mathcal{W}=\mathcal{W}_0 \gg \mathcal{W}_1 \gg \mathcal{W}_2 \gg \cdots .$$

Let c_k abbreviate "$\mathcal{W}_k(\mathsf{c})$". Since c is always the e_0 of the loops we consider, and since (3b) *adds* mass to even-subscripted edges, it follows that $c_{k+1}-c_k \geqslant 0$. Thus

3d: $$\sum_{k=1}^{\infty} |c_{k+1}-c_k| \quad \leqslant \quad \infty .$$

(Indeed, the sum is $\leqslant 1$, since each $c_k \in [0,1]$.)

For *each* edge e, note that the absolute value of $\mathcal{W}_k(\mathsf{e})-\mathcal{W}_{k-1}(\mathsf{e})$ is either c_k-c_{k-1} or is zero, depending on whether e was in the k^{th} loop or not. Since this is an absolutely-summable function of k, by (3d), we conclude that there is a well-defined non-negative limit function

$$\mathcal{U}(\cdot) \;:=\; \lim_{k\to\infty} \mathcal{W}_k(\cdot)$$

with $\mathcal{U} \ll \mathcal{W}$.

$\mathcal{U}$ <u>is</u> a weight-function. We seek[♡5] to establish (2a). Fix a vertex v. Since each loop involves at most two of v's edges we conclude that

$$\sum_{\mathsf{e}\in\mathsf{E}\langle v\rangle} \big|\mathcal{W}_{k+1}(\mathsf{e})-\mathcal{W}_k(\mathsf{e})\big| \;\leqslant\; 2\big|c_{k+1}-c_k\big| .$$

The RhS is a summable function of k. Thus we have convergence in the "little $\boldsymbol{\ell}^1$ norm" on $\boldsymbol{\ell}^1(\mathsf{E}\langle v\rangle)$ and may conclude that

$$\sum_{\mathsf{e}\in\mathsf{E}\langle v\rangle} \mathcal{U}(\mathsf{e}) \;=\; \lim_{k\to\infty} \sum_{\mathsf{e}\in\mathsf{E}\langle v\rangle} \mathcal{W}_k(\mathsf{e}) \;=\; \lim_{k\to\infty} 1 \;=\; 1 .$$

Deleting all zero-weight edges arranges that c is free of loops. Notice that this algorithm leaves the vertex set unchanged. ♦

Pf of the Main Thm.

Preserving the sum while freeing edges. Let v denote a fixed vertex incident on the edge c of the preceding section and let $\mathcal{F} \subset \mathsf{E}\langle v\rangle$ be any subset containing c. The k^{th} loop through c can have (loops have no repeated vertices) at most one edge in $\mathcal{F}$ other than c. If it does *not*, then $\sum_{\mathsf{e}\in\mathcal{F}} \mathcal{W}_k(\mathsf{e})$ *exceeds* $\sum_{\mathsf{e}\in\mathcal{F}} \mathcal{W}_{k-1}(\mathsf{e})$, otherwise they are equal. In any case we conclude that $\sum_{\mathsf{e}\in\mathcal{F}} \mathcal{W}_\infty(\mathsf{e})$ dominates $\sum_{\mathsf{e}\in\mathcal{F}} \mathcal{W}(\mathsf{e})$.

Now let $\mathcal{F}$ be a finite subset of $\mathsf{E}\langle v\rangle$ and free its edges one by one. Letting $s_{\mathcal{F}}(\cdot)$ denote the resulting weight-function, the preceding paragraph gives

3e: $$\sum_{\mathsf{e}\in\mathcal{F}} s_{\mathcal{F}}(\mathsf{e}) \;\geqslant\; \sum_{\mathsf{e}\in\mathcal{F}} \mathcal{W}(\mathsf{e})$$

(Actually, weight-function $s_{\mathcal{F}}$ depends on the *order* in which the edges were freed. We do not need this in the notation.)

[♡4] We can view $\mathcal{W}$ and $\mathcal{U}$ as *measures* on E, in which case $\mathcal{U} \ll \mathcal{W}$ is precisely "$\mathcal{U}$ is absolutely-cts w.r.t $\mathcal{W}$."

[♡5] A convergent sequence of weight-fncs need not be a weight-fnc: Consider a vertex v with edges $\mathsf{e}_1,\mathsf{e}_2,\ldots$, so that the k^{th} weight-fnc gives mass 1 to e_k, and zero to each other v-edge. The limit-fnc then gives total-weight zero to the v-edges.

Obtaining a graph without loops

We now use (3e) as a lemma. Let $u_1, u_2, \ldots, u_n, \ldots$ be a listing of vertices of the graph so that every vertex appear infinitely often in the list. Set $s_0(\cdot) := \mathcal{W}(\cdot)$.

Suppose, at stage $n-1$ we have chosen sets $\mathcal{F}_1, \ldots, \mathcal{F}_{n-1}$ with $\mathcal{F}_i$ a finite subset of $\mathsf{E}[u_i]$. And we have a weight-function s_{n-1}. Now choose a finite $\mathcal{F}_n \subset \mathsf{E}[u_n]$ sufficiently large that

$$\sum_{\mathsf{e} \in \mathcal{F}_n} s_{n-1}(\mathsf{e}) \;\geqslant\; 1 - \tfrac{1}{n}\,.$$

Apply operation (3e) to $\mathcal{F}_n$ to produce a new weight-function $s_n := s_{\mathcal{F}_n}$ satisfying: Each edge in $\mathcal{F}_n$ is free. And by (3e),

3f: $$\sum_{\mathsf{e} \in \mathcal{F}_n} s_n(\mathsf{e}) \;\geqslant\; 1 - \frac{1}{n}\,.$$

Do this inductively and arrange that $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathsf{E}$. Notice that once an edge has been freed, its weight does not change as later edges are freed. So there is a well-defined non-negative limit function $t(\cdot) := \lim_{n\to\infty} s_n(\cdot)$. And $t(\mathsf{e}) = s_n(\mathsf{e})$, for each $\mathsf{e} \in \mathcal{F}_n$. Thus (3f) holds for t replacing s_n.

Fix a vertex v. Since each vertex of the graph appears as infinitely many u_n,

$$\sum_{\mathsf{e} \in \mathsf{E}\langle v\rangle} t(\mathsf{e}) \;\geqslant\; \sup_n\,[1 - \tfrac{1}{n}] \;=\; 1$$

from (3f). Conversely, note that each s_n is non-negative on $\mathsf{E}\langle v\rangle$. Thus

$$\sum_{\mathsf{e} \in \mathsf{E}\langle v\rangle} t(\mathsf{e}) \;\leqslant\; \liminf_n \sum_{\mathsf{e} \in \mathsf{E}\langle v\rangle} s_n(\mathsf{e}) \;=\; 1$$

by Fatou's Lemma applied to counting measure on $\mathsf{E}\langle v\rangle$. Thus $t(\cdot)$ is a weight-function.

Seeing the trees in the forest

Delete all zero-weight edges. Our bipartite graph is now acyclic. For each edge e of weight 1, match its two vertices and delete them and e from the graph. Our acyclic bipartite graph now has the property that each edge has weight strictly between 0 and 1. So each component is a tree having no vertices of degree less-equal one. An application of the Forest Lemma completes the proof. ♦

Remark. (*March 30, 1992. This needs to be checked.*) In our bipartite graph, call the sets $\mathbb{B}, \mathbb{G}$ from the Main Theorem "boys" and "girls".

Assume now that the weights of edges are non-negative, that they sum to 1 at every girl, and sum to $\geqslant 1$ (possibly $+\infty$??) at every boy. We will marry-off the all the boys to some of the girls.

It seems to me that the "Iterating loop deletion" section preserves the total-weight at each vertex, and retains that edges have non-negative weight. Since every edge is incident on some girl, every edge has weight between 0 and 1.

Consider a boy with a weight=1 edge to a girl. Delete this boy and *every* girl he has a weight=1 edge to. Since such girls have *no* edges to any other boy, this does not foul up that

$$\mathcal{T}(\text{any boy}) \;\geqslant\; 1 \qquad \text{and} \qquad \mathcal{T}(\text{any girl}) = 1$$

where $\mathcal{T}(\cdot)$ denotes total weight at a vertex. Iterate this operation until there are no weight=1 edges.

Now I claim that the Forest Lemma applies. The total-weight at a root is $\geqslant 1$, but the maximum weight for every edge is $\leqslant 1$; and all weight$=$ 1 edges have been deleted. So there are no roots.

4: Corollary. *Suppose the boys all have degree b (a cardinality) and the girls all have degree $g \in \mathbb{Z}_+$ and $b \geqslant g$, THEN all the boys can be married-off.* ♢

Proof. The proof is to put weight $1/g$ on each edge. ♦

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