

Binomial Coefficients

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Entrance. (Use ‘ineq.’ for “inequality” and ‘coeff’ for “coefficient”. Use \asymp for “asymptotic to”; $f(n) \asymp g(n)$ means that $\frac{f(n)}{g(n)} \rightarrow 1$ as $n \nearrow \infty$.)

Stirling’s formula says that $n! \asymp \sqrt{2\pi n} \cdot [n/e]^n$. This implies that the central binomial coeff $\binom{2n}{n}$ is asymptotic to $4^n \cdot \frac{\text{Const}}{\sqrt{n}}$, where

$$\text{Const} = 1/\sqrt{\pi} \approx 0.564.$$

Overview. Fix a posint K for the rest of this note. Induction on $n \in [K .. \infty)$ will give

$$\begin{aligned} \alpha_n: \quad \binom{2n}{n} &\geq 4^n \cdot \frac{\mathbf{A}}{\sqrt{n}} \\ \beta_n: \quad \binom{2n}{n} &\leq 4^n \cdot \frac{\mathbf{B}}{\sqrt{T+n}}, \end{aligned}$$

for all $n \in [K .. \infty)$, once non-negative constants $\mathbf{B}, \mathbf{A}, T$ are chosen according to the following lemma. Necessarily, constants \mathbf{B} and \mathbf{A} will have to satisfy $\mathbf{B} \geq \frac{1}{\sqrt{\pi}} \geq \mathbf{A}$.

1.1: Central-coefficient lemma. Define A_K st. $\text{LhS}(\alpha_K)$ equals $\text{RhS}(\alpha_K)$, i.e, let

$$1.2: \quad \mathbf{A} := A_K := \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{K}.$$

For each $n \in [K .. \infty)$, then, inequality (α_n) holds.

Fix a real $T \geq \frac{K}{4K-1}$, then take \mathbf{B} so as to give equality in (β_K) . I.e, define

$$1.3: \quad \mathbf{B} := B_K := \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{T+K}.$$

Then inequality (β_n) holds for each $n \in [K .. \infty)$.

Lastly,

$$\gamma: \quad \binom{2n}{n} \geq \frac{1}{2n} \cdot 4^n,$$

for each $n \in \mathbb{Z}_+$. ◇

Preliminaries. Let \widehat{n} denote some unknown positive function of n , and let $\boxed{n} := 1/\widehat{n}$. Each of the three

bounds above is OTForm $\widehat{n} \cdot 4^n$. Thus each inequality has the form

$$\dagger(n): \quad \binom{2n}{n} \geq 4^n \cdot \widehat{n},$$

where relation \geq is either “ \geq ” or “ \leq ”.

Assume that we have verified (\dagger) for some base-case value of n . The induction step is $\dagger(n-1) \implies \dagger(n)$. Here is the algebra:

$$\begin{aligned} \binom{2n}{n} &= \binom{2n-2}{n-1} \cdot \frac{[2n-1] \cdot 2n}{n \cdot n} = 4 \cdot \binom{2n-2}{n-1} \cdot \frac{[2n-1]}{2n} \\ &\stackrel{\text{(by induction)}}{\geq} 4 \cdot [4^{n-1} \cdot \widehat{n-1}] \cdot \frac{[2n-1]}{2n} \\ &= 4^n \cdot \widehat{n-1} \cdot \frac{[2n-1]}{2n}. \end{aligned}$$

We want the RhS to $\geq 4^n \cdot \widehat{n}$, in order to continue the induction. So we wish to establish

$$\widehat{n-1} \cdot \frac{2n-1}{2n} \stackrel{?}{\geq} \widehat{n}.$$

Replace each \widehat{n} by $1/\boxed{n}$. This rewrites our goal as

$$\ddagger: \quad \frac{2n-1}{2n} \stackrel{?}{\geq} \frac{\boxed{n-1}}{\boxed{n}}. \quad \square$$

Proof of (γ) . Here, $\boxed{n} := 2n$, so the desired inequality (\ddagger) is

$$\frac{2n-1}{2n} \stackrel{?}{\geq} \frac{2[n-1]}{2n},$$

which is immediate. Finally, (γ) holds at $n=1$. ◆

Proof of (α) . Let $\boxed{n} := \sqrt{n}$; the multiplicative constant \mathbf{A} is irrelevant for (\ddagger) , the induction step. We wish to show that $\frac{2n-1}{2n} \geq \frac{\sqrt{n-1}}{\sqrt{n}}$. Squaring each side gives this equivalent ineq.

$$\left[1 - \frac{1/2}{n}\right]^2 \stackrel{?}{\geq} 1 - \frac{1}{n}.$$

But LhS^2 equals $\text{RhS} + [1/2n]^2$, so the inequality holds. ◆

Proof of (β). Set $\boxed{n} := \sqrt{T+n}$.

Inequality (\dagger) becomes $\frac{2n-1}{2n} \leq \frac{\sqrt{T+n-1}}{\sqrt{T+n}}$. This is equivalent (since $n \geq 1$, so $T+n-1 \geq 0$) to its square,

$$\left[1 - \frac{1}{2n}\right]^2 \leq 1 - \frac{1}{T+n}.$$

Squaring-out the LhS leads to $\frac{1}{4n^2} - \frac{1}{n} \leq -\frac{1}{T+n}$. So $\frac{4n-1}{4n^2} \geq \frac{1}{T+n}$, i.e. $T+n \geq \frac{4n^2}{4n-1}$. Thus $T \geq \frac{1}{4} \cdot \frac{4n}{4n-1}$. And since the RhS of this is decreasing, each

$$1.4: \quad T \geq \frac{K}{4K-1}$$

suffices for the induction to hold. ♦

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