Binomial Coefficients

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://www.math.ufl.edu/~squash/
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Entrance. (Use 'ineq.' for "inequality" and 'coeff' for "coefficient". Use $\asymp$ for "asymptotic to"; $f(n) \asymp g(n)$ means that $\frac{f(n)}{g(n)} \to 1$ as $n \nearrow \infty$.)

Stirling's formula says that $n! \asymp \sqrt{2\pi n} \cdot [n/e]^n$. This implies that the central binomal coeff $\binom{2n}{n}$ is asymptotic to $4^n \cdot \frac{\text{Const}}{\sqrt{n}}$, where

$$\text{Const} = 1/\sqrt{\pi} \approx 0.564 .$$

Overview. $\boxed{\text{Fix a posint } K}$ for the rest of this note. Induction on $n \in [K .. \infty)$ will give

$$\alpha_n\text{:} \quad \binom{2n}{n} \geqslant 4^n \cdot \frac{\mathbf{A}}{\sqrt{n}}$$
$$\beta_n\text{:} \quad \binom{2n}{n} \leqslant 4^n \cdot \frac{\mathbf{B}}{\sqrt{T+n}} ,$$

for all $n \in [K .. \infty)$, once non-negative constants $\mathbf{B}, \mathbf{A}, T$ are chosen according to the following lemma. Necessarily, constants $\mathbf{B}$ and $\mathbf{A}$ will have to satisfy $\mathbf{B} \geqslant \frac{1}{\sqrt{\pi}} \geqslant \mathbf{A}$.

1.1: Central-coefficient lemma. *Define A_K st. $\text{LhS}(\alpha_K)$ equals $\text{RhS}(\alpha_K)$, i.e, let*

1.2: $$\mathbf{A} := A_K := \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{K} .$$

For each $n \in [K .. \infty)$, then, inequality (α_n) holds.

Fix a real $T \geqslant \frac{K}{4K-1}$, then take $\mathbf{B}$ so as to give equality in (β_K). I.e, define

1.3: $$\mathbf{B} := B_K := \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{T+K} .$$

Then inequality (β_n) holds for each $n \in [K .. \infty)$.

Lastly,

$$\gamma\text{:} \quad \binom{2n}{n} \geqslant \frac{1}{2n} \cdot 4^n ,$$

for each $n \in \mathbb{Z}_+$. ◊

Preliminaries. Let $\widehat{n}$ denote some unknown *positive* function of n, and let $\boxed{n} := 1/\widehat{n}$. Each of the three bounds above is OTForm $\widehat{n} \cdot 4^n$. Thus each inequality has the form

$$\dagger(n)\text{:} \quad \binom{2n}{n} \gtrless 4^n \cdot \widehat{n} ,$$

where relation $\gtrless$ is either "$\geqslant$" or "$\leqslant$".

Assume that we have verified (†) for some base-case value of n. The induction step is $\dagger(n-1) \implies \dagger(n)$. Here is the algebra:

$$\binom{2n}{n} = \binom{2n-2}{n-1} \cdot \frac{[2n-1] \cdot 2n}{n \cdot n} = 4 \cdot \binom{2n-2}{n-1} \cdot \frac{[2n-1]}{2n}$$
$$\textit{(by induction)} \gtrless 4 \cdot \left[4^{n-1} \cdot \widehat{n-1}\right] \cdot \frac{[2n-1]}{2n}$$
$$= 4^n \cdot \widehat{n-1} \cdot \frac{[2n-1]}{2n} .$$

We want the RhS to $\gtrless 4^n \cdot \widehat{n}$, in order to continue the induction. So we wish to establish

$$\widehat{n-1} \cdot \frac{2n-1}{2n} \overset{?}{\gtrless} \widehat{n} .$$

Replace each $\widehat{n}$ by $1/\boxed{n}$. This rewrites our goal as

$$\ddagger\text{:} \quad \frac{2n-1}{2n} \overset{?}{\gtrless} \frac{\boxed{n-1}}{\boxed{n}} .$$ □

Proof of (γ). Here, $\boxed{n} := 2n$, so the desired inequality (‡) is

$$\frac{2n-1}{2n} \overset{?}{\geqslant} \frac{2[n-1]}{2n} ,$$

which is immediate. Finally, (γ) holds at $n=1$. ♦

Proof of (α). Let $\boxed{n} := \sqrt{n}$; the multiplicative constant $\mathbf{A}$ is irrelevant for (‡), the induction step. We wish to show that $\frac{2n-1}{2n} \geqslant \frac{\sqrt{n-1}}{\sqrt{n}}$. Squaring each side gives this equivalent ineq.

$$\left[1 - \frac{1/2}{n}\right]^2 \overset{?}{\geqslant} 1 - \frac{1}{n} .$$

But LhS^2 equals $\text{RhS} + [\frac{1/2}{n}]^2$, so the inequality holds. ♦

Proof of (β). Set $\boxed{n} := \sqrt{T+n}$.

Inequality (‡) becomes $\frac{2n-1}{2n} \leqslant \frac{\sqrt{T+n-1}}{\sqrt{T+n}}$. This is equivalent (since $n \geqslant 1$, so $T+n-1 \geqslant 0$) to its square,

$$\left[1 - \frac{1}{2n}\right]^2 \quad \leqslant \quad 1 - \frac{1}{T+n}\,.$$

Squaring-out the LhS leads to $\frac{1}{4n^2} - \frac{1}{n} \leqslant -\frac{1}{T+n}$. So $\frac{4n-1}{4n^2} \geqslant \frac{1}{T+n}$, i.e $T+n \geqslant \frac{4n^2}{4n-1}$. Thus $T \geqslant \frac{1}{4} \cdot \frac{4n}{4n-1}$. And since the RhS of this is decreasing, each

1.4: $$T \quad \geqslant \quad \frac{K}{4K-1}$$

suffices for the induction to hold. ♦

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