Binomial Coefficients

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23 September, 2017 (at 15:43)

Entrance. (Use ‘ineq.’ for “inequality” and ‘coeff’ for “coefficient”. Use $\asymp$ for “asymptotic to”; $f(n) \asymp g(n)$ means that $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$.)

Stirling's formula says that $n! \asymp \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$. This implies that the central binomial coeff $\binom{2n}{n}$ is asymptotic to $4^n \cdot \text{Const}/\sqrt{n}$, where

$$\text{Const} = 1/\sqrt{\pi} \approx 0.564.$$

Overview. (Fix a posint $K$) for the rest of this note. Induction on $n \in [K..\infty)$ will give

$$\begin{align*}
\alpha_n: & \quad \binom{2n}{n} \geq 4^n \cdot \frac{A}{\sqrt{n}} \\
\beta_n: & \quad \binom{2n}{n} \leq 4^n \cdot \frac{B}{\sqrt{T + n}},
\end{align*}$$

for all $n \in [K..\infty)$, once non-negative constants $B,A,T$ are chosen according to the following lemma. Necessarily, constants $B, A$ will have to satisfy $B \geq 1/\sqrt{\pi} \geq A$.

1.1: Central-coefficient lemma. Define $A_K$ st. LhS$(\alpha_K)$ equals RhS$(\alpha_K)$, i.e, let

$$A := A_K := \binom{2K}{K} / \sqrt{K}.$$

For each $n \in [K..\infty)$, then, inequality $(\alpha_n)$ holds. Fix a real $T \geq \frac{K}{K-1}$, then take $B$ so as to give equality in $(\beta_K)$. I.e, define

$$B := B_K := \binom{2K}{K} / \sqrt{T + K}.$$

Then inequality $(\beta_n)$ holds for each $n \in [K..\infty)$.

Lastly,

$$\gamma: \quad \binom{2n}{n} \geq \frac{1}{2n} \cdot 4^n,$$

for each $n \in \mathbb{Z}_+$. \hfill \blacklozenge

Preliminaries. Let $\hat{n}$ denote some unknown positive function of $n$, and let $[\hat{n}] := 1/\hat{n}$. Each of the three bounds above is OTForm $\hat{n} \cdot 4^n$. Thus each inequality has the form

$$\dagger(n): \quad \left(\frac{2n}{n}\right) \geq 4^n \cdot \hat{n},$$

where relation $\geq$ is either “$\geq$“ or “$\leq$“.

Assume that we have verified $(\dagger)$ for some base-case value of $n$. The induction step is $\dagger(n-1) \implies \dagger(n)$. Here is the algebra:

$$\begin{align*}
\frac{2n}{n} & = \left(\frac{2n-2}{n-1}\right) \cdot \frac{[2n-1] \cdot 2n}{n \cdot n} = 4 \cdot \left(\frac{2n-2}{n-1}\right) \cdot \frac{[2n-1]}{2n} \\
& \overset{\text{(by induction)}}{\geq} 4 \cdot \left[4^{n-1} \cdot \hat{n} \right] \cdot \frac{[2n-1]}{2n}.
\end{align*}$$

We want the RhS to $\geq 4^n \cdot \hat{n}$, in order to continue the induction. So we wish to establish

$$n-1 \cdot \frac{2n-1}{2n} \overset{\text{?}}{\geq} \hat{n}.$$

Replace each $\hat{n}$ by $1/\hat{n}$. This rewrites our goal as

$$\dagger: \quad \frac{2n-1}{2n} \overset{\text{?}}{\geq} \frac{[n-1]}{[n]}.$$ \hfill \blacklozenge

Proof of $(\gamma)$. Here, $[n] := 2n$, so the desired inequality $(\dagger)$ is

$$\frac{2n-1}{2n} \overset{?}{\geq} \frac{2n-1}{2n},$$

which is immediate. Finally, $(\gamma)$ holds at $n=1$. \hfill \blacklozenge

Proof of $(\alpha)$. Let $[\hat{n}] := \sqrt{\hat{n}}$; the multiplicative constant $A$ is irrelevant for $(\dagger)$, the induction step. We wish to show that $\frac{2n-1}{2n} \geq \sqrt{\hat{n}} / \sqrt{\hat{n}}$. Squaring each side gives this equivalent ineq.

$$\left[1 - \frac{1/2^2}{\hat{n}}\right] \overset{?}{\geq} 1 - \frac{1}{n}.$$ But LhS$^2$ equals RhS$+[1/2]^2$, so the inequality holds. \hfill \blacklozenge
Proof of \((\beta)\). Set \(\sqrt{n} := \sqrt{T+n}\).

Inequality \((\dagger)\) becomes \(\frac{2n-1}{2n} \leq \frac{\sqrt{T+n-1}}{\sqrt{T+n}}\). This is equivalent (since \(n \geq 1\), so \(T+n-1 \geq 0\)) to its square,

\[
\left[1 - \frac{1}{2n}\right]^2 \leq 1 - \frac{1}{T+n}.
\]

Squaring-out the LhS leads to \(\frac{1}{4n^2} - \frac{1}{n} \leq -\frac{1}{T+n}\). So \(\frac{4n-1}{4n^2} \geq \frac{1}{T+n}\), i.e \(T + n \geq \frac{4n^2}{4n-1}\). Thus \(T \geq \frac{1}{4} \cdot \frac{4n}{4n-1}\).

And since the RhS of this is decreasing, each

1.4:

\(T \geq \frac{K}{4K-1}\)

suffices for the induction to hold.

\(\diamondsuit\)