Differentiating a bilinear function

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/

3 November, 2015 (at *17:08*)

(Below, use ***VS*** for *vector space*, and ***IPS*** for *inner-product space*.)

Prolegomenon. In this pamphlet, all VSes are *real* VSes, $\mathbb{R}$-VSes, as I don't wish to discuss what $\mathbb{C}$-differentiation means.

The Product Rule from calculus states:

1a: *Suppose $f,g:\mathbb{R}\to\mathbb{R}$ are differentiable. Then so is their product, and*

$$[f\cdot g]' = [f\cdot g'] + [f'\cdot g].$$

It turns out this generalizes.

1b: *For an $\mathbb{R}$-IPS $\mathbf{V}$: Suppose $f,g:\mathbb{R}\to\mathbf{V}$ are diff'able fncs. Then so is $\langle f,g\rangle$, and*

$$\langle f,g\rangle' = \langle f,g'\rangle + \langle f',g\rangle.$$

Also, for Physics problems in 3-dim'al Euclidean space:

1c: *Suppose $f,g:\mathbb{R}\to\mathbb{R}^3$ are diff'able. Then so is $\langle f,g\rangle$. Moreover,*

$$\langle f\times g\rangle' = \langle f\times g'\rangle + \langle f'\times g\rangle.$$

All these raise the question (*not* "Beg the question", which means something different): What does it *mean* for a fnc $\mathbb{R}\to\mathbf{V}$ to be "differentiable"?

We suppose that $(\mathbf{V},\|\cdot\|)$ is a *normed VS*. For a fnc $f:\mathbb{R}\to\mathbf{V}$ at a point $\tau\in\mathbb{R}$, we can make sense of the difference-quotient

1d: $$\frac{f(\tau+h)-f(\tau)}{h}, \quad \text{for non-zero } h\in\mathbb{R}.$$

Sending $h\to 0$ might give a $\|\cdot\|$-limit; if so, we call the limit $f'(\tau)$.

Consider normed VSes $\mathbf{V},\mathbf{W},\mathbf{X}$, a fnc $\Omega:\mathbf{V}\times\mathbf{W}\to\mathbf{X}$, and a point $P := (\mathbf{v},\mathbf{w})$ in $\mathbf{V}\times\mathbf{W}$. Then Ω is **"**(jointly) continuous at P**"** if:

1e: *For all sequences $\mathbf{a}_n\to\mathbf{v}$ in $\mathbf{V}$, and $\mathbf{b}_n\to\mathbf{w}$ in $\mathbf{W}$, sequence*

$$\Omega(\mathbf{a}_n,\mathbf{b}_n) \text{ tends to } \Omega(\mathbf{v},\mathbf{w}) \text{ in } \mathbf{X}.$$

2: Product-rule Theorem. *Consider normed vector-spaces $\mathbf{V},\mathbf{W},\mathbf{X}$ and differentiable functions $\alpha:\mathbb{R}\to\mathbf{V}$ and $\beta:\mathbb{R}\to\mathbf{W}$. Suppose $[\![\cdot,\cdot]\!]$ is a bilinear map $\mathbf{V}\times\mathbf{W}\to\mathbf{X}$. If $[\![\cdot,\cdot]\!]$ is (jointly) continuous, then*

$$f(t) := [\![\alpha(t),\beta(t)]\!]$$

is differentiable, and

$$*: \quad [\![\alpha,\beta]\!]' = [\![\alpha,\beta']\!] + [\![\alpha',\beta]\!]. \qquad \Diamond$$

Pf. Fix $\tau\in\mathbb{R}$ and take a non-zero h. Then $f(\tau+h)-f(\tau)$ equals

$$[\![\alpha(\tau+h),\beta(\tau+h)]\!] - [\![\alpha(\tau+h),\beta(\tau)]\!] + [\![\alpha(\tau+h),\beta(\tau)]\!] - [\![\alpha(\tau),\beta(\tau)]\!].$$

Using linearity in each argument separately, $f(\tau+h) - f(\tau)$ equals

$$\left[\!\!\left[\alpha(\tau+h), \frac{\beta(\tau+h)-\beta(\tau)}{h}\right]\!\!\right] + \left[\!\!\left[\frac{\alpha(\tau+h)-\alpha(\tau)}{h}, \beta(\tau)\right]\!\!\right]$$

Sending $h\to 0$ yields $(*)$, courtesy the (joint) continuity. ♦

Filename: `Problems/Algebra/LinearAlg/bilinear-differentiability.latex`
As of: *Monday 28Nov2011.* Typeset: *3Nov2015* at *17:08.*