

## Differentiating a bilinear function

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(Below, use *VS* for vector space, and *IPS* for inner-product space.)

**Prolegomenon.** In this pamphlet, all VSes are *real* VSes,  $\mathbb{R}$ -VSes, as I don't wish to discuss what  $\mathbb{C}$ -differentiation means.

The Product Rule from calculus states:

1a: Suppose  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are differentiable. Then so is their product, and

$$[f \cdot g]' = [f \cdot g'] + [f' \cdot g].$$

It turns out this generalizes.

1b: For an  $\mathbb{R}$ -IPS  $\mathbf{V}$ : Suppose  $f, g: \mathbb{R} \rightarrow \mathbf{V}$  are diff'able fncs. Then so is  $\langle f, g \rangle$ , and

$$\langle f, g \rangle' = \langle f, g' \rangle + \langle f', g \rangle.$$

Also, for Physics problems in 3-dim'al Euclidean space:

1c: Suppose  $f, g: \mathbb{R} \rightarrow \mathbb{R}^3$  are diff'able. Then so is  $\langle f, g \rangle$ . Moreover,

$$\langle f \times g \rangle' = \langle f \times g' \rangle + \langle f' \times g \rangle.$$

All these raise the question (*not* "Beg the question", which means something different): What does it *mean* for a fnc  $\mathbb{R} \rightarrow \mathbf{V}$  to be "differentiable"?

We suppose that  $(\mathbf{V}, \|\cdot\|)$  is a *normed VS*. For a fnc  $f: \mathbb{R} \rightarrow \mathbf{V}$  at a point  $\tau \in \mathbb{R}$ , we can make sense of the difference-quotient

1d: 
$$\frac{f(\tau + h) - f(\tau)}{h}, \quad \text{for non-zero } h \in \mathbb{R}.$$

Sending  $h \rightarrow 0$  might give a  $\|\cdot\|$ -limit; if so, we call the limit  $f'(\tau)$ .

Consider normed VSes  $\mathbf{V}, \mathbf{W}, \mathbf{X}$ , a fnc  $\Omega: \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{X}$ , and a point  $P := (\mathbf{v}, \mathbf{w})$  in  $\mathbf{V} \times \mathbf{W}$ . Then  $\Omega$  is "(jointly) continuous at  $P$ " if:

1e: For all sequences  $\mathbf{a}_n \rightarrow \mathbf{v}$  in  $\mathbf{V}$ , and  $\mathbf{b}_n \rightarrow \mathbf{w}$  in  $\mathbf{W}$ , sequence

$$\Omega(\mathbf{a}_n, \mathbf{b}_n) \text{ tends to } \Omega(\mathbf{v}, \mathbf{w}) \text{ in } \mathbf{X}.$$

**2: Product-rule Theorem.** Consider normed vector-spaces  $\mathbf{V}, \mathbf{W}, \mathbf{X}$  and differentiable functions  $\alpha: \mathbb{R} \rightarrow \mathbf{V}$  and  $\beta: \mathbb{R} \rightarrow \mathbf{W}$ . Suppose  $\llbracket \cdot, \cdot \rrbracket$  is a bilinear map  $\mathbf{V} \times \mathbf{W} \rightarrow \mathbf{X}$ . If  $\llbracket \cdot, \cdot \rrbracket$  is (jointly) continuous, then

$$f(t) := \llbracket \alpha(t), \beta(t) \rrbracket$$

is differentiable, and

$$*: \llbracket \alpha, \beta \rrbracket' = \llbracket \alpha, \beta' \rrbracket + \llbracket \alpha', \beta \rrbracket. \quad \diamond$$

*Pf.* Fix  $\tau \in \mathbb{R}$  and take a non-zero  $h$ . Then  $f(\tau + h) - f(\tau)$  equals

$$\begin{aligned} & \llbracket \alpha(\tau + h), \beta(\tau + h) \rrbracket - \llbracket \alpha(\tau + h), \beta(\tau) \rrbracket \\ & + \llbracket \alpha(\tau + h), \beta(\tau) \rrbracket - \llbracket \alpha(\tau), \beta(\tau) \rrbracket. \end{aligned}$$

Using linearity in each argument separately,  $f(\tau + h) - f(\tau)$  equals

$$\left\llbracket \alpha(\tau + h), \frac{\beta(\tau + h) - \beta(\tau)}{h} \right\rrbracket + \left\llbracket \frac{\alpha(\tau + h) - \alpha(\tau)}{h}, \beta(\tau) \right\rrbracket$$

Sending  $h \rightarrow 0$  yields (\*), courtesy the (joint) continuity.  $\diamond$

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