Bertrand’s Postulate
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Background. Proofs are from Shoup, from Wikipedia and from my notes.

The superscript \( \otimes \). An inequality OTForm\[
\forall n: \; f(n) \leq 5^{\otimes} \cdot h(n).
\]

Similarly,\[
\forall n: \; f(n) \leq U \cdot h(n).
\]

Notice that \( U \) and \( L \) are quantified before \( n \).

Clumps. For \( p \) prime, let \( \text{Divlog}_p(1500) \) denote the maximum natnum \( L \) st. \( p^L \otimes 1500 \). Another notation for this is \( p^L \otimes 1500 \). So \( \text{Divlog}_5(1500) = 3 \).

For a non-zero integer \( B \), the \( \text{"p-clump of } B\)”, \( \text{Clm}(B) \), is the largest power of \( p \) which divides \( B \). So \( \text{Clm}_5(1500) \) is 125, and \( \text{Clm}_2(1500) = 4 \).

Evidently \( \text{Clm}_p(B) = p^\text{Divlog}_p(B); \) and \( B \)’s clumps multiplied-together make \( B \).

1: Lemma. Fix a prime \( p \) and natnum \( K \). Then
\[
\text{Divlog}_p(K!) = \sum_{j=1}^{\infty} \left\lfloor \frac{K}{p^j} \right\rfloor. \quad \text{(Exercise)} \quad \Diamond
\]

2: Prop’n. \( \forall \alpha \in \mathbb{R}: \; |2\alpha| - 2|\alpha| \) is zero or one. \quad \Diamond

We denote the set of prime numbers by \( \mathbb{P} \). Below, \( \text{\"p\} \) ranges over the prime numbers. All following definitions are for real \( x \), although usually \( x \) will be an integer.

First the “Product Of Primes”:
\[
\text{PrOP}(N) := \prod_{p: \; p \leq x} p.
\]

Its logarithm is the famous Chebyshev theta fnc:
\[
\vartheta(x) := \log(\text{PrOP}(N)) = \sum_{\rho: \; \rho \leq x} \log(\rho).
\]

Generalizing PrOP. When \( S \) is a set of reals, let \( \text{PrOP}(S) \) mean the product of the primes in \( S \).

3: PowFour Lemma. For each \( x \geq 1: \text{PrOP}(x) < 4^x \). in other words: \( \vartheta(x) < \log(4) \cdot x \), \quad \Diamond

Proof. WLOG, \( x \) is an integer \( N \).

\begin{align*}
| \text{Case: } N = 1 \rangle & : \; \text{PrOP}(1) = 1 < 4^1. \\
| \text{Case: } N = 2 \rangle & : \; \text{PrOP}(2) = 2 < 4^2. \\
| \text{Case: } N > 2 \text{ and } N \text{ is even} \rangle & : \; \text{PrOP}(N) = \text{PrOP}(N-1), \text{ since } N \text{ isn’t prime,} \\
& \quad < 4^{N-1}, \text{ by induction,} \\
& \text{which is less than } 4^N.
\end{align*}

\(| N > 2 \text{ and } N \text{ is odd} \rangle \text{Write } N := [2H + 1]. \text{ Induction gives (since } H+1 < N \text{) that}
\]
\[
\text{PrOP}([1 .. H+1]) < 4^{H+1},
\]

so our goal is to show that
\[3’: \quad \text{PrOP}((H+1 .. N]) \leq 4^H.\]

Flipping a coin \( N \) times, the number of coin-flip sequences is (letting \( j,k \) range over \( \mathbb{N} \))
\[
[1+1]^N = \sum_{j+k=N} \binom{N}{j,k} \geq \binom{N}{H,H+1} + \binom{N}{H+1,H} = 2 \cdot \binom{N}{H}.
\]

Divide by 2, then exchange sides, to get \( \binom{N}{H} \leq 4^H. \)

Each prime in \( (H+1 .. N) \) divides \( \binom{N}{H} \), so
\[
\text{PrOP}((H+1 .. N]) \otimes \binom{N}{H}.
\]

Since \( \binom{N}{H} \) is positive, \( \text{PrOP}((H+1 .. N]) \leq \binom{N}{H} \). Hence \( (?)’ \).

Prime-number Thm and related results
Use \( \pi(x) \) for the number of primes in \( [1, x] \). We’ll estimate it in terms of \( \frac{x}{\log(x)} \). Differentiating this latter gives:

4: The fnc \( x \mapsto \frac{x}{\log(x)} \) is strictly-increasing on the \( [e, \infty) \) interval.
5: Chebyshev’s Thm. For each posint \( n \geq 2 \):

5a: \[
\pi(n) \geq \frac{\log(2)}{2} \cdot \frac{n}{\log(n)}.
\]

Conversely, for each real \( U > \log(4) \):

5b: \[
\forall_{\text{large } x} : \quad \pi(x) \leq U \cdot \frac{x}{\log(x)}.
\]

\[\diamondsuit\]

Proof of (5a). Sound-bite: Produce a big integer \( B \) all of whose clumps are small. Since the clumps multiply to \( B \), and they are small, there must be many clumps. Hence many small primes divide \( B \). So many small primes exist. Thus \( \pi(x) \) must be big.

Even \( n \): Write \( 2N := n \). Let \( B := \binom{2N}{N} \). Easily

\[ B \geq 2^N. \]

Let \( T \) denote the number of distinct primes which divide \( B \); each such \( p \leq 2N \), so

\( \pi(2N) \geq T. \)

Lower-bounding \( T \). Evidently \( \text{Divlog}_p \left( \binom{2N}{N} \right) \) equals \( \text{Divlog}_p \left( \frac{2N}{p} \right) \). By (1), then,

\[
\text{Divlog}_p(B) = \sum_{j=1}^{\infty} \left[ \frac{2N}{p^j} \right] - 2 \sum_{j=1}^{\infty} \left[ \frac{N}{p^j} \right] = \sum_{j=1}^{L} \left[ \frac{2N}{p^j} \right] - 2 \left[ \frac{N}{p^j} \right],
\]

where \( L \) is \( \lfloor \log_p(2N) \rfloor \). By (2), each summand is either 1 or 0. Thus \( \log_p(2N) \geq \text{Divlog}_p(B) \). So

\[
2N \geq \text{Clm}_p(B),
\]

since \( p^{\text{Divlog}_p(B)} \) is \( \text{Clm}_p(B) \). Multiplying the \( B \)-clumps together gives \( B \), so \( 2N \geq B \). Hence \( 2N \geq 2^N \). Consequently \( T \cdot \log(2N) \geq \log(2) \cdot N \). Dividing yields (note \( N > 0 \), so \( \log(2N) \neq 0 \))

\[
T \geq \frac{\log(2)}{2} \cdot \frac{2N}{\log(2N)} \overset{\text{def}}{=} \frac{\log(2)}{2} \cdot \frac{n}{\log(n)}.
\]

Courtesy (6), this is the desired (5a).

Odd \( n \geq 3 \): Since \( n+1 \) is even, thus not prime,

\[
\pi(n) = \pi(n+1) \geq \frac{\log(2)}{2} \cdot \frac{n+1}{\log(n+1)}.
\]

Now use (4).

\[\diamondsuit\]

Proof of (5b). We will use Thm 10, below, being careful not to argue circularly.

By (10), there is a real, \( 1^+ \), and \( x_0 \) so that \( \forall x \geq x_0 : \pi(x) \leq 1^+ \cdot \frac{\theta(x)}{\log(x)} \). By the PowFour Lemma, then,

\[
\pi(x) \leq 1^+ \cdot \log(4) \cdot \frac{x}{\log(x)}.
\]

Chebyshev’s thm gives a growth rate on the \( n \)-th prime \( p_n \).

8: Theorem. Fix posreals \( L \leq U \) such that \( \forall_{\text{large } \ell} : \pi(\ell) \leq U \cdot \frac{\ell}{\log(\ell)} \).

Then \( \forall_{\text{large } n} : \)

\[
\left\lfloor \frac{1}{U} \right\rfloor \cdot n \log(n) \leq p_n \leq \left\lfloor \frac{1}{L} \right\rfloor \cdot n \log(n).
\]

\[\diamondsuit\]

Pf of (1\dagger). Fix a \( U > 1 \) with \( \forall_{\text{large } \ell} : \pi(\ell) \leq U \cdot \frac{\ell}{\log(\ell)} \).

Taking \( n \) sufficiently large, then,

\[
n \overset{\text{def}}{=} \pi(p_n) \leq U \cdot \frac{p_n}{\log(p_n)}. \]

Cross-multiplying gives \( \frac{1}{U} \cdot n \log(p_n) \leq p_n \).

But \( n \leq p_n \), so \( \log(n) \leq \log(p_n) \). Thus

\[
\frac{1}{U} \cdot n \log(n) \leq p_n. \]

\[\diamondsuit\]

Proof of (2\dagger). Suppose \( \forall_{\text{large } \ell} : \pi(\ell) \geq \frac{1}{5} \cdot \frac{\ell}{\log(\ell)} \).

We want to establish (2\dagger), with the constant being 5\dagger.

Define \( K_n \) by \( p_n = K_n \cdot n \log(n) \). Let \( S \) be the set of \( n \) with \( \left\lfloor K_n \right\rfloor \geq 5.001 \). FTSC, suppose \( S \) is infinite.

For large \( n \in S \), then \( \frac{1}{5} \cdot p_n \leq \log(p_n) \overset{\text{def}}{=} n \). So

\[
\frac{1}{5} \leq \frac{\log(K_n \cdot n \log(n))}{K_n \log(n)} = \frac{\log(K_n \log(n))}{K_n \log(n)} + \frac{1}{K_n}.
\]

Note that \( [K_n \log(n)] \to \infty \), as \( n \to \infty \), since \( \{K_n\}_{1}^{\infty} \) is bounded below, and \( \log(n) \to \infty \). Apply to each side \( \limsup_{n \to \infty} \left( \frac{1}{K_n} \right) \) but only for \( n \in S \), to obtain that

\[
\frac{1}{5} \leq \limsup_{n \to \infty} \frac{1}{K_n} \overset{\text{note}}{=} \frac{1}{5.001}. \]

This contradiction shows \( S \) must have been finite! \[\diamondsuit\]

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9: Lemma. Fix a positive \( \delta < 1 \). Then \( x^\delta = o\left(\frac{x}{\log(x)}\right) \). Consequently,

9*: \[ x^\delta = o(\pi(x)). \]

Proof. Use l'Hôpital’s rule. For (9*), note that (5a) implies \( \frac{x}{\log(x)} = O(\pi(x)) \).

10: Asymptotic \( \pi, \vartheta \) Thm. Indeed,

1. \( \pi(x) \geq \vartheta(x) / \log(x), \) for all \( x > 1 \).

2. \( \pi(x) \asymp \vartheta(x) / \log(x), \) as \( x \to \infty \).

Proof. When \( p \leq x \), necessarily \( \log(p) \leq \log(x) \). So

\[ \vartheta(x) \leq \sum_{\rho \in (1..x]} \log(x) = \log(x) \cdot \pi(x). \]

Because of (10.1), ISTFix a posreal \( \varepsilon \) and show

\[ [1 + \varepsilon] \frac{\vartheta(x)}{\log(x)} \geq [1 - o(1)] \cdot \pi(x), \]

to establish the (10.2) asymptotics. Rewritten, our goal is

\[ [1 + \varepsilon] \frac{\vartheta(x)}{\log(x)} \geq \pi(x) - o(\pi(x)). \]

So fix a positive \( \delta < 1 \) and set \( L := x^\delta \). Thus

\[ \vartheta(x) \geq \sum_{\rho \in (L..x]} \log(L) = \delta \log(x) \cdot [\pi(x) - \pi(L)]. \]

Hence \( \frac{\vartheta(x)}{\log(x)} \geq \pi(x) - \pi(L) \). Therefore, we need but show that \( \pi(L) = o(\pi(x)) \). But \( \pi(L) \leq L = x^\delta \). And (9*) is our knight in shining armor.

11: Coro. There is a positive constant \( C \) so that

\( \forall_{\text{large } n}: \quad C \cdot n \leq \vartheta(n). \)

Proof. Combine (10.2) with (5a).

The \( n \)th harmonic number is \( H_n := \sum_{j=1}^{n} \frac{1}{j} \), for \( n \) a posint. Easily,

\( \vdash: \forall n: \quad H_n \geq H_{n-1} \geq \log(n) \geq H_n - 1. \)

\( \vdash: \forall x > 0: \quad x \geq \log(1 + x). \)

Euler proved that \( \sum_{p \leq x} \frac{1}{p} = \infty \). His argument essentially shows (12), below.

12: Thm. For \( N \) a posint: \( \sum_{p \leq N} \frac{1}{p} \geq \loglog(N). \)

Hence, \( \sum_{p \leq N} \frac{1}{p} \geq \loglog(N) - O(1). \)

Proof. Each \( n \leq N \) is some product of \( p_j^{\varepsilon_j} \), over primes \( p_j \leq N \). So \( \frac{1}{n} \) has form \( \prod_{p \leq N} \frac{1}{p^{\varepsilon_j}} \). Thus

\[ H_N \leq \prod_{p \leq N} \left[ 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots \right]. \]

And \( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots = \frac{1}{1 - \frac{1}{p}} = 1 + \frac{1}{p - 1} \). By (\( \vdash \)), then,

\( \loglog(N) \leq \log(H_N) \leq \sum_{p \leq N} \log(1 + \frac{1}{p - 1}). \) Hence

\( \loglog(N) \leq \sum_{p \leq N} \left[ \frac{1}{p - 1} \right]. \)

Shoup’s proof of Bertrand’s postulate

Let \( T_n := \pi(2n) - \pi(n) \). The PNT suggests that \( T_n \approx \frac{n}{\log(2N)} \). We will show this weaker stat.

13: Bertrand’s Density Postulate. For each posint \( N \):

13\( \vdash \): \( T_N \geq \frac{1}{3} \cdot \frac{N}{\log(2N)}. \)

Rem. It will suffice to produce a constant \( U > \frac{1}{3} \) st.

13\( \vdash \): \( T_N \geq U \cdot \frac{N}{\log(2N)} - o\left(\frac{N}{\log(2N)}\right), \)
then verify (13\( \vdash \)) for finitely many values of \( N \).

Proof. We use notation from (5a) and its proof.

Each prime \( p \in B \) produces a clump \( Clm_p := p^{\text{Divlog}_p(B)} \). Given an interval \( J \subset (1..2N) \), let \( J \) be
the product of the \( p \)-clumps over all \( p \in J \). We will show that (proof is currently omitted)

\[
(1 - \sqrt{2N}) \leq [2N]^{\sqrt{2N}} ;
\]

\[
(\sqrt{2N} - \frac{2}{3}N) \leq 4[\frac{2}{3}N] ;
\]

\[
(\frac{2}{3}N - N) = 1 ;
\]

\[
(N - 2N) \leq [2N]^T .
\]

But \( B \) is the product of its clumps, so

\[
[2N]^T \cdot [2N]^{\sqrt{2N}} \cdot 4[\frac{2}{3}N] \geq B .
\]

A simple induction shows that \( \binom{2N}{n} \geq \frac{1}{2n} \cdot 4^n \). Thus

\[
[2N]^{T+1+\sqrt{2N}} \geq 4[\frac{1}{3}N] .
\]

So \([T + 1+\sqrt{2N}] \cdot \log(2N) \geq \log(4) \cdot \frac{1}{3}N \). Thus

\[
T \geq \log(4) \cdot \frac{N}{3 \log(2N)} - \lfloor 1+\sqrt{2N} \rfloor.
\]

And this is what we needed in (13†).

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**Logarithmic Integral**

Following Shoup, define \(^{\odot}\)

\[
\text{Li}(x) := \int_2^x \frac{1}{\log(t)} dt .
\]

Let’s use L’Hôpital’s rule to show that

\[14: \quad \text{Li}(x) \asymp \frac{x}{\log(x)} .\]

Abbrev \( \log(x) \) by \( L \). So \( \frac{d}{dx} \left(\frac{x}{\log(x)}\right) = \frac{1 - \frac{x}{L}}{L^2} = \frac{1}{L} - \frac{1}{L^2} \). Therefore

\[
\frac{[\log(x)]'}{[\text{Li}(x)]'} = \frac{\frac{1}{L} - \frac{1}{L^2}}{\frac{1}{L}} = 1 - \frac{1}{L} .
\]

And \( 1 - \frac{1}{L} \to 0 \) as \( x \to \infty \). Hence \( \text{Li}(x) \) yields (14).

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\(^{\odot}\)Wikipedia calls this version the “Offset logarithmic integral”, and uses \( \text{Li}_0 \) for its “logarithmic integral”.

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**Erdős’ proof of Bertrand’s postulate**

Assume there is a CEX: an integer \( N \geq 2 \) such that there is no prime number in \((N, 2N)\).

If \( N \in [2, 2048) \), then one of the prime numbers 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259 and 2503 (each being less than twice its predecessor), call it \( p \), will satisfy \( N < p < 2N \). Therefore WLOG \( N \geq 2048 \).

**Proof, when \( N \geq 2048 \).** Note that

\[
4^N = [1 + 1]^{2N} = \sum_{k=0}^{2N} \left(\begin{array}{c}2N \\ k\end{array}\right) .
\]

Since \( \left(\begin{array}{c}2N \\ N\end{array}\right) \) is the largest term in the sum, we have that

\[
\frac{4^N}{2N+1} \leq \left(\begin{array}{c}2N \\ N\end{array}\right) .
\]

Define \( R := R(p, N) \) to be highest integer \( x \), such that \( p^x \) divides \( \left(\begin{array}{c}2N \\ N\end{array}\right) \). Applying (1) to \( K := 2N \) and \( K := N \) yields

\[
R = \text{Divlog}_p([2N]!) - 2 \cdot \text{Divlog}_p(N!)
\]

\[
= \sum_{j=1}^\infty \left\lfloor \frac{2N}{p^j} \right\rfloor - 2 \sum_{j=1}^\infty \left\lfloor \frac{N}{p^j} \right\rfloor .
\]

But each term

\[
\left\lfloor \frac{2N}{p^j} \right\rfloor - 2 \left\lfloor \frac{N}{p^j} \right\rfloor
\]

can either be 0 (when \( N \) mod 1 < \( \frac{1}{2} \)) or 1 (when \( N \) mod 1 \( \geq \) \( \frac{1}{2} \)). Furthermore, all terms with

\[
j \geq \left\lceil \frac{\log(2N)}{\log(p)} \right\rceil
\]

are 0. Therefore

\[
R \leq \left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor .
\]

and we get:

\[
p^R = \exp(R \cdot \log(p)) \leq \exp \left( \frac{\log(2N)}{\log(p)} \log(p) \right) \leq 2N .
\]
For each $p > \sqrt{2N}$, necessarily
\[
\left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor \leq 1
\]
or
\[
\mathcal{R} = \left\lfloor \frac{2N}{p} \right\rfloor - 2 \left\lfloor \frac{N}{p} \right\rfloor.
\]
Remark that $\binom{2N}{N}$ has no prime factors $p$ such that:

- $2N < p$, because $2N$ is the largest factor.
- $N < p \leq 2N$, because we assumed there is no such prime number.
- $\frac{2N}{3} < p \leq N$, because (since $N \geq 5$) which gives us
\[
\mathcal{R} = \left\lfloor \frac{2N}{p} \right\rfloor - 2 \left\lfloor \frac{N}{p} \right\rfloor = 2 - 2 = 0.
\]

Each prime factor of $\binom{2N}{N}$ is therefore not larger than $\frac{2N}{3}$.

Note that $\binom{2N}{N}$ has at most one factor of every prime $p > \sqrt{2N}$. As $p^\mathcal{R} \leq 2N$, the product of $p^\mathcal{R}$ over all other primes is at most $[2N]^{\sqrt{2N}}$. Since $\binom{2N}{N}$ is the product of $p^\mathcal{R}$ over all primes $p$, we get that
\[
\frac{4^N}{2N + 1} \leq \binom{2N}{N} \leq [2N]^{\sqrt{2N}} \cdot \prod_{p \in \mathbb{P}} p
\]
\[
= [2N]^{\sqrt{2N}} \cdot e^{\vartheta\left(\frac{2N}{3}\right)}.
\]

Using our lemma, $\vartheta(N) < N \cdot \log(4)$:
\[
\frac{4^N}{2N + 1} \leq [2N]^{\sqrt{2N}} \cdot 4^{\frac{2N}{3}}
\]
Since we have $[2N + 1] < [2N]^2$, automatically
\[
4^{\frac{N}{3}} \leq [2N]^{2 + \sqrt{2N}}.
\]
Also $2 \leq \frac{\sqrt{2N}}{3}$ (since $N \geq 18$): Consequently,
\[
4^{\frac{N}{3}} \leq [2N]^{\frac{2}{3}\sqrt{2N}}.
\]
Taking logarithms produces
\[
\sqrt{2N} \cdot \log(2) \leq 4 \cdot \log(2N).
\]