Bertrand's Postulate

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Background. Proofs are from Shoup, from Wikipedia and from my notes.

The superscript '⊕'. An inequality OTForm $\boxed{\forall_{\text{large}} n\colon\ f(n) \leqslant 5^{\oplus} \cdot h(n)}$ means

$$\forall \mathbf{U} > 5, \quad \forall_{\text{large}} n\colon \quad f(n) \leqslant \mathbf{U} \cdot h(n).$$

Similarly, $\boxed{\forall_{\text{large}} n\colon\ f(n) \geqslant [\tfrac{1}{3}]^{\ominus} \cdot h(n)}$ means

$$\forall_{\text{positive}} \mathbf{L} < \tfrac{1}{3}, \quad \forall_{\text{large}} n\colon \quad f(n) \geqslant \mathbf{L} \cdot h(n).$$

Notice that $\mathbf{U}$ and $\mathbf{L}$ are quantified ***before*** n.

Clumps. For p prime, let $\text{Divlog}_p(1500)$ denote the maximum natnum L st. $p^L \bullet\!| \ 1500$. Another notation for this is $p^L \bullet\!\| \ 1500$. So $\text{Divlog}_5(1500) = 3$.

For a non-zero integer B, the "**p-clump of B**", $\text{Clm}_p(B)$, is the largest power of p which divides B. So $\text{Clm}_5(1500)$ is 125, and $\text{Clm}_2(1500) = 4$.

Evidently $\text{Clm}_p(B) = p^{\text{Divlog}_p(B)}$; and B's clumps multiplied-together make B.

1: Lemma. *Fix a prime p and natnum K. Then*

$$\text{Divlog}_p(K!) = \sum_{j=1}^{\infty} \left\lfloor \frac{K}{p^j} \right\rfloor. \quad (\textit{Exercise}) \ \Diamond$$

2: Prop'n. $\forall \alpha \in \mathbb{R}$: $\lfloor 2\alpha \rfloor - 2\lfloor \alpha \rfloor$ *is zero or one.* ◊

We denote the set of prime numbers by $\mathbb{P}$. Below, "p" ranges over the prime numbers. All following definitions are for real x, although usually x will be an integer.

First the "**P**roduct **O**f **P**rimes",

$$\text{PrOP}(N) := \prod_{p\colon\ p \leqslant x} p.$$

Its logarithm is the famous ***Chebyshev theta fnc***:

$$\boldsymbol{\vartheta}(x) := \log(\text{PrOP}(N)) = \sum_{p\colon\ p \leqslant x} \log(p).$$

Generalizing PrOP. When S is a set of reals, let $\text{PrOP}(S)$ mean the product of the primes in S.

3: PowFour Lemma. *For each $x \geqslant 1$: $\text{PrOP}(x) < 4^x$. in other words: $\boldsymbol{\vartheta}(x) < \log(4) \cdot x$,* ◊

Proof. WLOG, x is an integer N.

$\boxed{\text{Case: } N = 1}$: $\text{PrOP}(1) = 1 < 4^1$.
$\boxed{\text{Case: } N = 2}$: $\text{PrOP}(2) = 2 < 4^2$.
$\boxed{\text{Case: } N > 2 \text{ and } N \text{ is even}}$

$$\begin{aligned} \text{PrOP}(N) &= \text{PrOP}(N-1), \text{ since } N \text{ isn't prime}, \\ &< 4^{N-1}, \text{ by induction}, \end{aligned}$$

which is less than 4^N.

$\boxed{N > 2 \text{ and } N \text{ is odd}}$ Write $N := [2H+1]$. Induction gives (since $H+1 < N$) that

$$\text{PrOP}\Big([1 \mathrel{..} H+1]\Big) < 4^{H+1},$$

so our goal is to show that

$$3'\colon \quad \text{PrOP}\Big((H+1 \mathrel{..} N]\Big) \overset{?}{\leqslant} 4^H.$$

Flipping a coin N times, the number of coin-flip sequences is (letting j, k range over $\mathbb{N}$)

$$\begin{aligned} [1+1]^N &= \sum_{j+k=N} \binom{N}{j,\ k} \\ &\geqslant \binom{N}{H,\ H+1} + \binom{N}{H+1,\ H} = 2 \cdot \binom{N}{H}. \end{aligned}$$

Divide by 2, then exchange sides, to get $\binom{N}{H} \leqslant 4^H$.

Each prime in $(H+1 \mathrel{..} N]$ divides $\binom{N}{H}$, so

$$\text{PrOP}\Big((H+1 \mathrel{..} N]\Big) \bullet\!| \ \binom{N}{H}.$$

Since $\binom{N}{H}$ is positive, $\text{PrOP}\Big((H+1 \mathrel{..} N]\Big) \leqslant \binom{N}{H}$. Hence (**??**′). ♦

Prime-number Thm and related results

Use $\pi(x)$ for the number of primes in $[1, x]$. We'll estimate it in terms of $\frac{x}{\log(x)}$. Differentiating this latter gives:

4: *The fnc $x \mapsto \frac{x}{\log(x)}$ is strictly-increasing on the $[\mathsf{e}, \infty)$ interval.*

5: Chebyshev's Theorem. *For each posint $n \geqslant 2$:*

5a: $$\pi(n) \;\geqslant\; \frac{\log(2)}{2} \cdot \frac{n}{\log(n)}\,.$$

Conversely, for each real $\mathbf{U} > \log(4)$:

5b: $$\forall_{\text{large}}\, x: \quad \pi(x) \;\leqslant\; \mathbf{U} \cdot \frac{x}{\log(x)}\,. \qquad \diamondsuit$$

Proof of (5a). *Sound-bite*: *Produce a big integer B all of whose clumps are small. Since the clumps multiply to B, and they are small, there must be many clumps. Hence many small primes divide B. So many small primes exist. Thus $\pi(x)$ must be big.*

Even n: Write $2N := n$. Let $B := \binom{2N}{N,N}$. Easily
$$B \;\geqslant\; 2^N\,.$$

Let $\mathbf{T}$ denote the number of distinct primes which divide B; each such $p \leqslant 2N$, so

6: $$\pi(2N) \;\geqslant\; \mathbf{T}\,.$$

***Lower-bnding* $\mathbf{T}$.** Evidently $\operatorname{Divlog}_p\!\left(\binom{2N}{N,N}\right)$ equals $\operatorname{Divlog}_p([2N]!) - 2\cdot\operatorname{Divlog}_p(N!)$. By (1), then,
$$\begin{aligned}\operatorname{Divlog}_p(B) &= \sum_{j=1}^{\infty}\left\lfloor\frac{2N}{p^j}\right\rfloor - 2\sum_{j=1}^{\infty}\left\lfloor\frac{N}{p^j}\right\rfloor \\ &= \sum_{j=1}^{L}\left[\left\lfloor\tfrac{2N}{p^j}\right\rfloor - 2\left\lfloor\tfrac{N}{p^j}\right\rfloor\right],\end{aligned}$$

where L is $\lfloor\log_p(2N)\rfloor$. By (2), each summand is either 1 or 0. Thus $\log_p(2N) \geqslant \operatorname{Divlog}_p(B)$. So

7: $$2N \;\geqslant\; \operatorname{Clm}_p(B)\,,$$

since $p^{\operatorname{Divlog}_p(B)}$ is $\operatorname{Clm}_p(B)$. Multiplying the B-clumps together gives B, so $[2N]^{\mathbf{T}} \geqslant B$. Hence $[2N]^{\mathbf{T}} \geqslant 2^N$. Consequently $\mathbf{T}\cdot\log(2N) \geqslant \log(2)\cdot N$. Dividing yields (note $N > 0$, so $\log(2N) \neq 0$)
$$\mathbf{T} \;\geqslant\; \frac{\log(2)}{2}\cdot\frac{2N}{\log(2N)} \;\stackrel{\text{def}}{=\!=}\; \frac{\log(2)}{2}\cdot\frac{n}{\log(n)}\,.$$

Courtesy (6), this is the desired (5a).

Odd $n \geqslant 3$: Since $n{+}1$ is even, thus not prime,
$$\pi(n) \;=\; \pi(n{+}1) \;\geqslant\; \frac{\log(2)}{2}\cdot\frac{n{+}1}{\log(n{+}1)}\,.$$

Now use (4). ♦

Proof of (5b). We will use Thm 10, below, being careful not to argue circularly.

By (10), there is a real, 1^+, and x_0 so that $\forall x \geqslant x_0$: $\pi(x) \leqslant 1^+\cdot\frac{\vartheta(x)}{\log(x)}$. By the PowFour Lemma, then,
$$\pi(x) \;\leqslant\; 1^+\cdot\log(4)\cdot\frac{x}{\log(x)}\,. \qquad ♦$$

Chebyshev's thm gives a growth rate on the n^{th}-prime p_n.

8: Theorem. *Fix posreals $\mathbf{L} \leqslant \mathbf{U}$ such that $\forall_{\text{large}}\,\ell$:*
$$\mathbf{L}^{\ominus}\cdot\frac{\ell}{\log(\ell)} \;\stackrel{1\dagger}{\leqslant}\; \pi(\ell) \;\stackrel{2\dagger}{\leqslant}\; \mathbf{U}^{\oplus}\cdot\frac{\ell}{\log(\ell)}\,.$$

Then $\forall_{\text{large}}\, n$:
$$\left[\tfrac{1}{\mathbf{U}}\right]^{\ominus}\cdot n\log(n) \;\stackrel{1\ddagger}{\leqslant}\; p_n \;\stackrel{2\ddagger}{\leqslant}\; \left[\tfrac{1}{\mathbf{L}}\right]^{\oplus}\cdot n\log(n)\,. \quad \diamondsuit$$

Pf of (1‡). Fix a $U > \mathbf{U}$ with $\forall_{\text{large}}\,\ell: \pi(\ell) \leqslant U\cdot\frac{\ell}{\log(\ell)}$. Taking n sufficiently large, then,
$$n \;\stackrel{\text{def}}{=\!=}\; \pi(p_n) \;\leqslant\; U\cdot\tfrac{p_n}{\log(p_n)}\,.$$

Cross-multiplying gives $\frac{1}{U}\cdot n\log(p_n) \leqslant p_n$. But $n \leqslant p_n$, so $\log(n) \leqslant \log(p_n)$. Thus
$$\tfrac{1}{U}\cdot n\log(n) \;\leqslant\; p_n\,. \qquad ♦$$

Proof of (2‡). Suppose $\forall_{\text{large}}\,\ell: \pi(\ell) \geqslant \frac{1}{5}\cdot\frac{\ell}{\log(\ell)}$. We want to establish (2‡), with the constant being $5^{\oplus}$.

Define K_n by $p_n = K_n\cdot n\log(n)$. Let S be the set of n with $\boxed{K_n \geqslant 5.001}$. FTSOC, suppose S is infinite.

For large $n{\in}S$, then, $\frac{1}{5}\cdot\frac{p_n}{\log(p_n)} \leqslant \pi(p_n) \stackrel{\text{def}}{=\!=} n$. So
$$\tfrac{1}{5} \;\leqslant\; \frac{\log\bigl(K_n\cdot n\log(n)\bigr)}{K_n\log(n)} \;=\; \frac{\log\bigl(K_n\log(n)\bigr)}{K_n\log(n)} + \frac{1}{K_n}\,.$$

Note that $[K_n\log(n)] \to \infty$, as $n \to \infty$, since $\{K_n\}_1^{\infty}$ is bnded below, and $\log(n) \to \infty$. Apply to each side $\limsup_{n\to\infty}$, $\boxed{\textit{but only for } n \in S}$, to obtain that
$$\tfrac{1}{5} \;\leqslant\; \limsup_{\substack{n\to\infty\\ n\in S}} \tfrac{1}{K_n} \;\stackrel{\text{note}}{\leqslant}\; \tfrac{1}{5.001}\,.$$

This contradiction shows S must have been *finite*! ♦

9: Lemma. *Fix a positive $\delta < 1$. Then $x^\delta = \mathsf{o}(\frac{x}{\log(x)})$. Consequently,*

$$9*: \qquad x^\delta = \mathsf{o}(\pi(x)). \qquad \diamond$$

Proof. Use l'Hôpital's rule. For (9∗), note that (5a) implies $\frac{x}{\log(x)} = \mathsf{O}(\pi(x))$. ♦

10: Asymptotic π, ϑ Thm. *Indeed,*

$$.1: \qquad \pi(x) \geqslant \frac{\vartheta(x)}{\log(x)}, \quad \text{for all } x > 1.$$

$$.2: \qquad \pi(x) \asymp \frac{\vartheta(x)}{\log(x)}, \quad \text{as } x \to \infty. \qquad \diamond$$

Proof. When $p \leqslant x$, necessarily $\log(p) \leqslant \log(x)$. So

$$\vartheta(x) \leqslant \sum_{p \in (1..x]} \log(x) = \log(x) \cdot \pi(x).$$

Because of (10.1), ISTFix a posreal ε and show

$$[1+\varepsilon]\frac{\vartheta(x)}{\log(x)} \overset{?}{\geqslant} [1 - \mathsf{o}(1)] \cdot \pi(x),$$

to establish the (10.2) asymptotics. Rewritten, our goal is

$$[1+\varepsilon] \cdot \frac{\vartheta(x)}{\log(x)} \overset{?}{\geqslant} \pi(x) - \mathsf{o}(\pi(x)).$$

So fix a positive $\delta<1$ and set $\mathbf{L} := x^\delta$. Thus

$$\vartheta(x) \geqslant \sum_{p \in (\mathbf{L}..x]} \log(\mathbf{L}) = \delta\log(x) \cdot [\pi(x) - \pi(\mathbf{L})].$$

Hence $\frac{1}{\delta} \cdot \frac{\vartheta(x)}{\log(x)} \geqslant \pi(x) - \pi(\mathbf{L})$. Therefore, we need but show that $\pi(\mathbf{L})$ is $\mathsf{o}(\pi(x))$. But $\pi(\mathbf{L}) \leqslant \mathbf{L} = x^\delta$. And (9∗) is our knight in shining armor. ♦

11: Coro. *There is a positive constant C so that*

$$\forall_{\text{large}}\, n: \qquad C \cdot n \leqslant \vartheta(n). \qquad \diamond$$

Proof. Combine (10.2) with (5a). ♦

The n^{th} ***harmonic number*** is $H_n := \sum_{j=1}^n \frac{1}{j}$, for n a posint. Easily,

$$\dagger: \quad \forall n: \qquad H_n \geqslant H_{n-1} \geqslant \log(n) \geqslant H_n - 1.$$

$$\ddagger: \quad \forall x > 0: \qquad x \geqslant \log(1+x).$$

Euler proved that $\sum_p \frac{1}{p} = \infty$. His argument essentially shows (12), below.

12: Thm. *For N a posint: $\sum_{p:\, p \leqslant N} \frac{1}{p-1} \geqslant \operatorname{loglog}(N)$. Hence, $\sum_{p \leqslant N} \frac{1}{p} \geqslant \operatorname{loglog}(N) - \mathsf{O}(1)$.* ◊

Proof. Each $n \leqslant N$ is some product of ${p_j}^{e_j}$, over primes $p_j \leqslant N$. So $\frac{1}{n}$ has form $\prod_{p \leqslant N} \frac{1}{p^{e(n)}}$. Thus

$$H_N \leqslant \prod_{p \leqslant N} \left[1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\right].$$

And $1 + \frac{1}{p} + \frac{1}{p^2} + \ldots = \frac{1}{1-\frac{1}{p}} = 1 + \frac{1}{p-1}$. By (†), then, $\operatorname{loglog}(N) \leqslant \log(H_N) \leqslant \sum_{p \leqslant N} \log(1 + \frac{1}{p-1})$. Hence

$$\operatorname{loglog}(N) \overset{\text{by } (\ddagger)}{\leqslant} \sum_{p \leqslant N} \left[\frac{1}{p-1}\right]. \qquad \blacklozenge$$

Shoup's proof of Bertrand's postulate

Let $\mathbf{T}_n := \pi(2n) - \pi(n)$. The PNT suggests that $\mathbf{T}_n \approx \frac{n}{\log(2n)}$. We will show this weaker stmt.

13: Bertrand's Density Postulate. *For each posint N:*

$$13\dagger: \qquad \mathbf{T}_N \geqslant \frac{1}{3} \cdot \frac{N}{\log(2N)}. \qquad \diamond$$

Rem. It will suffice to produce a constant $\mathbf{U} > \frac{1}{3}$ st.

$$13\ddagger: \qquad \mathbf{T}_N \geqslant \mathbf{U} \cdot \frac{N}{\log(2N)} - \mathsf{o}(\tfrac{N}{\log(2N)}),$$

then verify (13†) for finitely many values of N. □

Proof. We use notation from (5a) and its proof.

Each prime $p \bullet\!| B$ produces a clump $\mathrm{Clm}_p := p^{\mathrm{Divlog}_p(B)}$. Given an interval $J \subset (1..2N]$, let $\overline{J}$ be the product of the *p-clumps* over all $p \in J$. We will show that (*proof is currently omitted*)

$$\begin{aligned}
\overline{(1 \,..\, \sqrt{2N}]} &\leqslant [2N]^{\sqrt{2N}}\,; \\
\overline{(\sqrt{2N} \,..\, \tfrac{2}{3}N]} &\leqslant 4^{[\frac{2}{3}N]}\,; \\
\overline{(\tfrac{2}{3}N \,..\, N]} &= 1\,; \\
\overline{(N \,..\, 2N]} &\leqslant [2N]^{\mathbf{T}}\,.
\end{aligned}$$

But B is the product of its clumps, so

$$[2N]^{\mathbf{T}} \cdot [2N]^{\sqrt{2N}} \cdot 4^{[\frac{2}{3}N]} \quad\geqslant\quad B\,.$$

A simple induction shows that $\binom{2n}{n} \geqslant \frac{1}{2n} \cdot 4^n$. Thus

$$[2N]^{\mathbf{T}+1+\sqrt{2N}} \quad\geqslant\quad 4^{[\frac{1}{3}N]}\,.$$

So $[\mathbf{T} + 1+\sqrt{2N}] \cdot \log(2N) \geqslant \log(4) \cdot \frac{1}{3}N$. Thus

$$\mathbf{T} \;\geqslant\; \log(4) \cdot \frac{1}{3}\frac{N}{\log(2N)} \;-\; [1+\sqrt{2N}]\,.$$

And this is what we needed in (13‡). ♦

Logarithmic Integral

Following Shoup, define[♡1]

$$\mathrm{Li}(x) \;:=\; \int_2^x \frac{1}{\log(t)}\,\mathrm{d}t\,.$$

Let's use L'Hôpital's rule to show that

14: $$\mathrm{Li}(x) \;\asymp\; \frac{x}{\log(x)}\,.$$

Abbrev $\log(x)$ by L. So $\frac{\mathrm{d}}{\mathrm{d}x}\big(\frac{x}{L}\big) = \frac{1 \cdot L - x \cdot \frac{1}{x}}{L^2} = \frac{1}{L} - \frac{1}{L^2}$. Therefore

$$\frac{\big[\frac{x}{\log(x)}\big]'}{[\mathrm{Li}(x)]'} \;=\; \frac{\frac{1}{L} - \frac{1}{L^2}}{\frac{1}{L}} \;=\; 1 - \frac{1}{L}\,.$$

And $1 - \frac{1}{L} \to 0$ as $x \to \infty$. Hence l'Hôpital's yields (14).

[♡1] Wikipedia calls this version the "Offset logarithmic integral", and uses $\int_0^\infty$ for its "logarithmic integral".

Erdős' proof of Bertrand's postulate

Assume there is a CEX: an integer $N \geqslant 2$ such that there is <u>no</u> prime number in $(N \,..\, 2N)$.

If $N \in [2 \,..\, 2048)$, then one of the prime numbers 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259 and 2503 (each being less than twice its predecessor), call it p, will satisfy $N < p < 2N$. Therefore WLOG $N \geqslant 2048$.

Proof, when $N \geqslant 2048$. Note that

$$4^N \;=\; [1+1]^{2N} \;=\; \sum_{k=0}^{2N} \binom{2N}{k}\,.$$

Since $\binom{2N}{N}$ is the largest term in the sum, we have that

$$\frac{4^N}{2N+1} \;\leqslant\; \binom{2N}{N}\,.$$

Define $\mathcal{R} := \mathcal{R}(p, N)$ to be highest integer x, such that p^x divides $\binom{2N}{N}$. Applying (1) to $K := 2N$ and $K := N$ yields

$$\begin{aligned}
\mathcal{R} &= \mathrm{Divlog}_p([2N]!) \;-\; 2 \cdot \mathrm{Divlog}_p(N!) \\
&= \sum\nolimits_{j=1}^{\infty} \left\lfloor \frac{2N}{p^j} \right\rfloor \;-\; 2\sum\nolimits_{j=1}^{\infty} \left\lfloor \frac{N}{p^j} \right\rfloor \\
&= \sum\nolimits_{j=1}^{\infty} \Big[\big\lfloor \tfrac{2N}{p^j} \big\rfloor - 2\big\lfloor \tfrac{N}{p^j} \big\rfloor \Big]\,.
\end{aligned}$$

But each term

$$\left\lfloor \frac{2N}{p^j} \right\rfloor - 2\left\lfloor \frac{N}{p^j} \right\rfloor$$

can either be 0 (when $\frac{N}{p^j} \bmod 1 < \frac{1}{2}$) or 1 (when $\frac{N}{p^j} \bmod 1 \geqslant \frac{1}{2}$). Furthermore, all terms with

$$j > \left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor$$

are 0. Therefore

$$\mathcal{R} \leqslant \left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor,$$

and we get:

$$\begin{aligned}
p^{\mathcal{R}} &= \exp(\mathcal{R} \cdot \log(p)) \\
&\leqslant \exp\left(\left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor \log(p) \right) \;\leqslant\; 2N\,.
\end{aligned}$$

For each $p > \sqrt{2N}$, necessarily

$$\left\lfloor \frac{\log(2N)}{\log(p)} \right\rfloor \leqslant 1$$

or

$$\mathcal{R} = \left\lfloor \frac{2N}{p} \right\rfloor - 2\left\lfloor \frac{N}{p} \right\rfloor .$$

Remark that $\binom{2N}{N}$ has no prime factors p such that:

- $2N < p$, because $2N$ is the largest factor.
- $N < p \leqslant 2N$, because we assumed there is no such prime number.
- $\frac{2N}{3} < p \leqslant N$, because (since $N \geqslant 5$) which gives us

$$\mathcal{R} = \left\lfloor \frac{2N}{p} \right\rfloor - 2\left\lfloor \frac{N}{p} \right\rfloor = 2 - 2 = 0 .$$

Each prime factor of $\binom{2N}{N}$ is therefore not larger than $\frac{2N}{3}$.

Note that $\binom{2N}{N}$ has at most one factor of every prime $p > \sqrt{2N}$. As $p^{\mathcal{R}} \leqslant 2N$, the product of $p^{\mathcal{R}}$ over all other primes is at most $[2N]^{\sqrt{2N}}$. Since $\binom{2N}{N}$ is the product of $p^{\mathcal{R}}$ over all primes p, we get that

$$\frac{4^N}{2N+1} \leqslant \binom{2N}{N} \leqslant [2N]^{\sqrt{2N}} \cdot \prod_{p \in \mathbb{P}}^{\frac{2N}{3}} p$$
$$= [2N]^{\sqrt{2N}} \cdot \mathrm{e}^{\boldsymbol{\vartheta}(\frac{2N}{3})} .$$

Using our lemma, $\boldsymbol{\vartheta}(N) < N \cdot \log(4)$:

$$\frac{4^N}{2N+1} \leqslant [2N]^{\sqrt{2N}} \cdot 4^{\frac{2N}{3}}$$

Since we have $[2N+1] < [2N]^2$, automatically

$$4^{\frac{N}{3}} \leqslant [2N]^{2+\sqrt{2N}} .$$

Also $2 \leqslant \frac{\sqrt{2N}}{3}$ (since $N \geqslant 18$): Consequently,

$$4^{\frac{N}{3}} \leqslant [2N]^{\frac{4}{3}\sqrt{2N}} .$$

Taking logarithms produces

$$\sqrt{2N} \cdot \log(2) \leqslant 4 \cdot \log(2N) .$$

Substituting 2^{2t} for $2N$:

$$\frac{2^t}{t} \leqslant 8 .$$

This gives us $t < 6$ and the contradiction that

$$N = \frac{2^{2t}}{2} < \frac{2^{2\cdot 6}}{2} \overset{\text{note}}{=\!=} 2048 .$$

Thus no counterexample to the postulate is possible.♦

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