Sets and Logic
MHF3202 2787

Home-B

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Tuesday, 16Oct., 2018

Due **BoC, Monday, 22Oct2018**, wATMP!
Please *fill-in* every *blank* on this sheet. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

B1: *Show no work. Simply fill-in each blank on the problem-sheet.*

a On a K-element set, the number of reflexive symmetric binrels is $2^{\binom{K}{2}} = 2^{\frac{[K-1]K}{2}}$.

On a 4-set, there are 15 many equiv.relations.

URL https://en.wikipedia.org/wiki/Partition_of_a_set has a picture. Let P_n be the # of ptns having precisely n nv-atoms. Then $P_1 = 1$, $P_2 = \binom{4}{1} + \frac{1}{2}\binom{4}{2} = 7$, $P_3 = \binom{4}{2} = 6$, $P_4 = 1$.

b On $\mathbb{Z}_+$, write $x \,\$\, y$ IFF $\text{GCD}(x,y) \geqslant 2$. So \$ is

Circle

Transitive: *T* *F*. **Symm.**: *T* *F*. **Reflex.**: *T* *F*.

CEX to transitivity: $\text{GCD}(2,6) \geqslant 2$ and $\text{GCD}(6,3) \geqslant 2$, yet $\text{GCD}(2,3) \ngeqslant 2$.

On $\mathbb{Z}$, say that $x \nabla y$ IFF $x - y < 1$. Then ∇ is:

Trans.: *T* *F*. **Symm.**: *T* *F*. **Reflex.**: *T* *F*.

Since x,y are *integers*, inequality $x - y < 1$ is equivalent to $x - y \leqslant 0$, i.e, to $x \leqslant y$. And $\leqslant$ is indeed transitive.

c On $\Omega := [1..29] \times [1..29]$, define binary-relation $\mathbf{C}$ by: $(x,\alpha)\,\mathbf{C}\,(y,\beta)$ IFF $x \cdot \beta \equiv_{30} y \cdot \alpha$. Statement "*Relation* $\mathbf{C}$ *is an **equivalence relation***" is: *T* *F*

Crossmult Soln: Relation $\mathbf{C}$ is not transitive, due to $\mathbb{Z}_{30}$ having non-trivial zero-divisors. For a CEX, note $(5,15)\,\mathbf{C}\,(3,3)$ and $(3,3)\,\mathbf{C}\,(1,1)$, yet $(5,15)$ is not $\mathbf{C}$-related to $(1,1)$.

For the two essay questions, carefully TYPE, *double-or-triple–spaced, grammatical solns.*

B2: Consider board $\mathbf{B}_N := 2 \times N$; so [board] is $\mathbf{B}_5$. Use T_N for the number of tilings of $\mathbf{B}_N$ by 1×1 ("1-minos") and 2×1, 1×2 ("dominos"). Evidently $T_0 = 1$ and $T_1 = 2$.

I PROVE: *Each natnum N satisfies*

$$*: \quad T_{N+2} = T_N + 2\sum_{j=0}^{N+1} T_j .$$

In addition to your essay, show your ideas in pictures.

II Derive a Fibonacci-like CCLRR

$$T_{N+3} = 3T_{N+2} + T_{N+1} - T_N .$$

So $T_7 =$ 2356, and $T_n = \alpha A^n + \beta B^n + \gamma C^n$, for some numbers α,β,γ, where A,B,C are roots of polynomial

$$f(x) = x^3 - 3x^2 - x + 1 .$$

For large n, then, $T_n \approx \alpha A^n$, where [decimal approximation] $A \approx 3.21432$ and $\alpha \approx 0.66458$.

Soln II:. In $(*)$, replacing N by $N+1$ yields

$$\begin{aligned}
T_{N+3} &= T_{N+1} + \Big[2\sum_{j=0}^{N+2} T_j\Big] + T_N - T_N \\
&= T_{N+1} + 2T_{N+2} - T_N + T_N + \Big[2\sum_{j=0}^{N+1} T_j\Big] \\
&\overset{\text{by }(*)}{=\!=} T_{N+1} + 2T_{N+2} - T_N + T_{N+2} \\
&= 3T_{N+2} + T_{N+1} - T_N .
\end{aligned}$$

Since $T_0 = 1$, $T_1 = 2$ and [courtesy $(*)$] $T_2 = 7$, this recurrence produces https://oeis.org/A030186 .

B3: In our Velleman text, solve problem #**12**P.277. Let $\mathbf{E}_n$ be the equilateral triangle with side-length 2^n. This $\mathbf{E}_n$ can be tiled in an obvious way by 4^n many little-triangles [copies of $\mathbf{E}_0$]; see picture P.277. The "***punctured*** $\mathbf{E}_n$", written $\widetilde{\mathbf{E}_n}$, has its topmost copy of $\mathbf{E}_0$ removed.

A ***(trape)zoid***, $\mathbf{T}$, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up [picture P.277]. [A ***zoid-tiling*** allows all three rotations of $\mathbf{T}$.]

i PROVE: *For each n, board $\widetilde{\mathbf{E}_n}$ admits a zoid-tiling.*

ii Let Δ_k be the equilateral triangle of sidelength k; so $\mathbf{E}_n$ is Δ_{2^n}. Triangle Δ_k comprises k^2 little-triangles.

For what values of k does Δ_k admit a zoid-tiling?

For which k does $\widetilde{\Delta_k}$ admit a zoid-tiling?

iii An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n \times 2^n$ square board, comprising 4^n ***squaries*** (little squares). Have $\widetilde{\mathbf{S}_n}$ be the board with one corner squarie removed. Velleman inductively shows, pp.272-275, that each $\widetilde{\mathbf{S}_n}$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, V. proves that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n \times 2^n \times 2^n$ cube, $\widetilde{\mathbf{C}_n}$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

What is the 3-dimensional analog of an Lmino? Calling it a "**3-*mino***", how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

iv Generalize to K-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n \times \overset{K}{\ldots} \times 2^n$ cube, having $[2^n]^K = 2^{nK}$ many K-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your K-**mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a K-mino–tiling.*

See: The Zoids (PDF) pamphlet on our webpage.

B1:	___ ___	95pts
B2:	___ ___	80pts
B3:	___ ___ ___	155pts
Total:	___ ___ ___	330pts