Sets and Logic
MHF3202 2787

Home-B

Prof. JLF King
Tuesday, 16Oct., 2018

Due **BoC, Monday, 22Oct2018**, wATMP!
Please *fill-in* every *blank* ____ on this sheet. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

B1: *Show no work. Simply fill-in each blank on the problem-sheet.*

a On a K-element set, the number of reflexive symmetric binrels is ____.

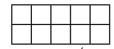
On a 4-set, there are ____ many equiv.relations.

b On $\mathbb{Z}_+$, write $x \mathbin{\$} y$ IFF $\text{GCD}(x,y) \geqslant 2$. So \$ is *Circle*
Transitive: *T F*. **Symm.**: *T F*. **Reflex.**: *T F*.

On $\mathbb{Z}$, say that $x \nabla y$ IFF $x - y < 1$. Then ∇ is:
Trans.: *T F*. **Symm.**: *T F*. **Reflex.**: *T F*.

c On $\Omega := [1..29] \times [1..29]$, define binary-relation $\mathbf{C}$ by: $(x,\alpha)\,\mathbf{C}\,(y,\beta)$ IFF $x \cdot \beta \equiv_{30} y \cdot \alpha$. Statement "*Relation* **C** *is an* ***equivalence relation***" is: *T F*

For the two essay questions, carefully Type, *double-or-triple–spaced, grammatical solns.*

B2: Consider board $\mathbf{B}_N := 2 \times N$; so  is $\mathbf{B}_5$. Use T_N for the number of tilings of $\mathbf{B}_N$ by $1{\times}1$ ("1-minos") and $2{\times}1$, $1{\times}2$ ("dominos"). Evidently $T_0 = 1$ and $T_1 = 2$.

I Prove: *Each natnum N satisfies*

$$*: \quad T_{N+2} = T_N + 2\sum_{j=0}^{N+1} T_j\,.$$

In addition to your essay, show your ideas in pictures.

II Derive a Fibonacci-like CCLRR

$T_{N+3} =$ ____.

So $T_7 =$ ____, and $T_n = \alpha A^n + \beta B^n + \gamma C^n$, for some numbers α,β,γ, where A,B,C are roots of polynomial

$f(x) =$ ____.

For large n, then, $T_n \approx \alpha A^n$, where [decimal approximation] $A \approx$ ____ and $\alpha \approx$ ____.

B3: In our Velleman text, solve problem **#12**$^{P.277}$. Let $\mathbf{E}_n$ be the equilateral triangle with side-length 2^n. This $\mathbf{E}_n$ can be tiled in an obvious way by 4^n many little-triangles [copies of $\mathbf{E}_0$]; see picture P.277. The **"*punctured* $\mathbf{E}_n$"**, written $\widetilde{\mathbf{E}}_n$, has its topmost copy of $\mathbf{E}_0$ removed.

A ***(trape)zoid***, $\mathbf{T}$, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up [picture P.277]. [A ***zoid-tiling*** allows all three rotations of $\mathbf{T}$.]

i PROVE: *For each n, board $\widetilde{\mathbf{E}}_n$ admits a zoid-tiling.*

ii Let Δ_k be the equilateral triangle of sidelength k; so $\mathbf{E}_n$ is Δ_{2^n}. Triangle Δ_k comprises k^2 little-triangles.

For what values of k does Δ_k admit a zoid-tiling?

For which k does $\widetilde{\Delta}_k$ admit a zoid-tiling?

iii An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n \times 2^n$ square board, comprising 4^n ***squaries*** (little squares). Have $\widetilde{\mathbf{S}}_n$ be the board with one corner squarie removed. Velleman inductively shows, pp.272-275, that each $\widetilde{\mathbf{S}}_n$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, V. proves that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n \times 2^n \times 2^n$ cube, $\widetilde{\mathbf{C}}_n$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

What is the 3-dimensional analog of an Lmino? Calling it a **"3-*mino*"**, how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

iv Generalize to K-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n \times \overset{K}{\cdots} \times 2^n$ cube, having $[2^n]^K = 2^{nK}$ many K-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your K-**mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a K-mino–tiling.*

B1:	___ ___	95pts
B2:	___ ___	80pts
B3:	___ ___ ___	155pts
Total:	___ ___ ___	330pts