DynSys MTG 6401 | **Home-B** | Prof. JLF King 31Aug2015

Intro. Due, no later than **11AM,Monday, 14Dec**, slid completely under my office door, LIT402.

A family $\mathcal{A} \subset \mathcal{P}(X)$ is an ***algebra*** if $\mathcal{A}$ is sealed under complement, pairwise union and pairwise intersection.

For a map $f: X \to X$, use f^n for $f^{\circ n} = f \circ f \circ \overset{n}{\cdots} \circ f$.

B1: Exhibit an $(S: Y, \mathcal{Y}, \nu)$, where Y is a nv-cpt metric space with Borel field $\mathcal{Y}$ and non-atomic prob.meas ν, and S is a bi-mpt homeomorphism. Construct S so that $S \times S$ is both topologically conjugate, and isomorphic, to S.

Produce an example where S is ergodic; you may quote theorems from class. [*Hint:* What are the ergodic-multipliers? What properties are sealed under projective limits?]

B2: Voila $f: X \circlearrowleft$, an isometry of a complete metric space $(X, \lceil\cdot,\cdot\rceil)$. Suppose we have a point $\mathbf{z} \in X$ whose orbit $\{\mathbf{z}_n\}_{n \in \mathbb{Z}}$ is dense, where $\mathbf{z}_n := f^n(\mathbf{z})$.

a Construct (with proof, natch') an abelian group operation $\boxplus$ on X and element $\boldsymbol{\alpha} \in X$ so that $\mathbf{z}$ is the $\boxplus$-identity and: $\forall x \in X$: $f(x) = x \boxplus \boldsymbol{\alpha}$.

[*Hint*: Use Cauchy seqs to define the addition. Show that your defn is indep of the Cauchy seqs used. **Note:** Without completeness, the result can *fail*. Would you be so kind as to give me such an example?]

[*Sugg*: For $x, y \in X$ define "$x \boxplus y$" by establishing: There exist sequences $\vec{k}$ and $\vec{\ell}$ with $T^{k_j}(\mathbf{z}) \to x$ and $T^{\ell_j}(\mathbf{z}) \to y$, as $j \nearrow \infty$, and $T^{k_j + \ell_j}(\mathbf{z})$ converges to some point, s. Argue that s is independent of $\vec{k}$ and $\vec{\ell}$. What is $\boxminus y$? Etc.]

b (If you wish, YMAssume X cpt.) No longer an isometry, suppose now that f is ***equicontinuous***. This means that the family $\{f^n \mid n \in \mathbb{Z}\}$ is an equicontinuous family, i.e given $x \in X$ and ε there exists $\delta = \delta(x, \varepsilon)$ such that

$$\forall y \in X, \forall n \in \mathbb{Z}: \quad \lceil x, y \rceil < \delta \ \Rightarrow\ \lceil f^n x, f^n y \rceil < \varepsilon\,.$$

Show that the conclusion of part (a) holds nonetheless. [*Hint:* Reduce to part (a). Does your reduction preserve: *Density of* $\mathcal{O}_f(\mathbf{z})$? *Completeness?*]

B3: Let $X := \mathbb{Z}$. A set $E \subset \mathbb{Z}$ has "[upper/lower] density β" if the limsup/liminf limit as $n \nearrow \infty$ of $\frac{1}{n}\#(E \cap [1..n])$ equals β. The set is **"*eventually-periodic* with period p"** if for all large positive n:

$$n \in E \implies n + p \in E\,.$$

i Prove that $\mathcal{A}$, the collection of eventually-periodic sets, is an *algebra* of sets, and that $\mu :=$ Density is a **finitely-additive probability "measure"** (a ***FAMe***) on $\mathcal{A}$.

ii Produce a bijection $T: \mathbb{Z} \circlearrowleft$ that preserves both $\mathcal{A}$ and $\mu()$, and produce a positive-mass set $B \in \mathcal{A}$ so that: $\forall x \in X$, the Cesàro averages

$$\frac{1}{N} \sum_{i \in [0..N)} \mathbf{1}_B(T^i x)$$

have limsup$\neq$liminf. Conclude that the conclusion of Birkhoff's thm can *fail* for **FAMes**.

Bonus For each $E \in \mathcal{A}$, its mass $\mu(E)$ is rational. Can you produce a CEX space and map, but where the measure takes on all values in $[0, 1]$?

End of Home-B

B1:	___ ___	95pts
B2:	___ ___ ___	115pts
B3:	___ ___ ___	115pts
Total:	___ ___ ___	325pts

Please PRINT your name and ordinal. Ta:

Ord: ....

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ....