Staple!

Team: B

Abstract Algebra
MAS4301 09B1

Home-B

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07Feb2008

Hello. Essays violate the CHECKLIST at *Grade Peril*!
Due **BoC, Monday, 28Oct2019**, wATMP!

Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0 \neq$ *Empty-word.*.

Let F and R be the ***flip*** and ***rotation*** in the dihedral group $\mathbb{D}_N$, with $\mathtt{F}^2 = \mathbf{e}$, $\mathtt{R}^N = \mathbf{e}$ and $\mathtt{RFRF} = \mathbf{e}$. Use $\mathtt{R}^j$ and $\mathtt{R}^j\mathtt{F}$ as the standard form of each element in $\mathbb{D}_N$.

Fill-in *all* blanks (*handwriting; don't bother to type*) on this sheet **including** the blanks for the essay questions!

B1: Show no work.

___ ___ **a** $N := \varphi(100) =$ ________ . So $\varphi(N) =$ ________ .
EFT says that $3^{1621} \equiv_N$ ________ $\in [0 .. N)$. Hence (by EFT) last two digits of $7^{[3^{1621}]}$ are ________ .

___ ___ **b** With $N := 212960000 = 5^4 \cdot 2^8 \cdot 11^3$, the longest *chain* of subgroups

$$\{\varepsilon\} \subsetneqq \langle g_1 \rangle \subsetneqq \langle g_2 \rangle \subsetneqq \ldots \subsetneqq \langle g_K \rangle = \mathbb{Y}_N$$

has $K =$ ________ . [*Hint:* No OBO!]
And $\mathbb{Y}_N$ has ________ subgroups.

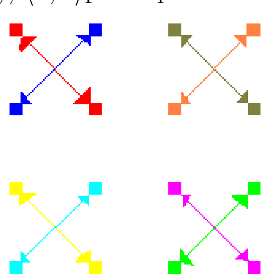
___ ___ **c** Let $V_K := G \times G \times \overset{K}{\cdots} \times G$, where $G := (\mathbb{Z}_2, +)$. Define $A_K := \mathrm{Aut}(V_K)$. Then $A_2 \cong \mathbb{D}_N$, where $N =$ ________ .
As a decreasing product of integers,
$|A_3| =$ ________ and $|A_4| =$ ________ .

___ ___ **d** Perm $\pi := [6, 7, 8, 1, 2, 3, 4, 5]$ has $\mathrm{Sgn}(\pi) = +\mathbf{1}\ -\mathbf{1}$.

For the essay questions, carefully TYPE*, double-spaced, grammatical solns.*

B2: The 4×4 TTT (TicTacToe) board is $\mathbb{B} := [1..4] \times [1..4]$; sixteen ***cells***. Let Γ denote the TTT-automorphism group; the set of self-bijections of $\mathbb{B}$ which preserve all ten TTTs. So $\mathtt{R}, \mathtt{F} \in \Gamma$, where R rotates $\mathbb{B}$ by 90° CCW, and F flips $\mathbb{B}$ about its vertical axis. Evidently, $\langle \mathtt{R}, \mathtt{F} \rangle_\Gamma \cong \mathbb{D}_4$.

A less obvious TTT-automorphism is the ***swizzle***, S: It exchanges each corner square with the central square that it (diagonally) touches; and it does The Right Thing on the edge squares.



i Easily, $\mathtt{S} \leftrightarrows \mathtt{R}$ and $\mathtt{S} \leftrightarrows \mathtt{F}$, hence each element of subgroup $\Lambda := \langle \mathtt{S}, \mathtt{F}, \mathtt{R} \rangle$ can be written in form $\mathtt{S}^b \mathtt{R}^c \mathtt{F}^d$ for integers b, c, d. So $|\Lambda| =$ ________ ; prove this.

Draw a "16-dot picture" (4×4 dots, with arrows), for each element $\alpha \in \Lambda$. However, use the same picture for α rotated or flipped about any line, or for α^{-1} (rotated or flipped). *Label* each picture with *all* the automorphisms that it describes. The total number of labels should equal your $|\Lambda|$.

ii Find a TTT-aut T which is **not** in the Λ subgp. Write the commutation relations between T and each of $\{\mathtt{S}, \mathtt{R}, \mathtt{F}\}$. Prove that each aut α can be written as $\alpha = \mathtt{T}^{a_1} \mathtt{S}^{a_2} \mathtt{R}^{a_3} \mathtt{F}^{a_4}$, with $a_i \in \mathbb{Z}$. For element $\beta = \mathtt{T}^{b_1} \mathtt{S}^{b_2} \mathtt{R}^{b_3} \mathtt{F}^{b_4}$, give an explicit multiplication rule showing how to compute the exponents $\{c_i\}_{i=1}^4$ of $\beta\alpha = \mathtt{T}^{c_1} \mathtt{S}^{c_2} \mathtt{R}^{c_3} \mathtt{F}^{c_4}$.

Prove that $\langle \mathtt{T}, \mathtt{S}, \mathtt{R}, \mathtt{F} \rangle$ is **all** of Γ. Thus $|\Gamma| =$ ________ .

Draw all the new labeled 16-dot pictures for $\Gamma \setminus \Lambda$.

iii Find a set of *involutions* which generates Γ. Compute (with proof) the center of Γ; what is its order?

iv With $\mathbf{u} \in \mathbb{B}$ the upper-LH corner of $\mathbb{B}$, define its stabilizer $\Upsilon := \mathrm{Stab}_\Gamma(\mathbf{u})$. With proof, compute $|\Upsilon| =$ ________ .
The number of Υ-orbits is ________ .

B3: Prove that $S := (\mathbb{Q}, +, 0)$ is *not* isomorphic to a proper subgroup of itself.

End of Home-B

B1:	___ ___ ___	110pts
B2:	___ ___ ___	150pts
B3:	___ ___	65pts
Total:	___ ___ ___	325pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

________________________________ Ord: ____

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