

Combinatorics
MAD4203 3214

Home-B

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10Oct2017

Due **BoC, Monday, 16Oct2017**, Please *fill-in* every *blank* on this sheet. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

Use $\mathcal{S}(N, K)$ for *Stirling# of 2^{nd} kind*.

A perm of cycle-signature $[7^3, 4^2, 1^8]$ has three 7-cycles, two 4-cycles, and eight fixed-pts. Use **CN** for “cycle notation”, and **CCN** for “canonical CN”.

B1: *Henceforth, show no work. Simply fill-in each blank on the problem-sheet.*

a For a tuple such as $\vec{c} := (5, 1, 2)$, let $\binom{8}{\vec{c}}$ mean multinom-coeff $\binom{8}{5,1,2}$. Fixing natnums J and L , let Γ denote the set of length- L compositions of J . What shorthand do we have from class

for this sum? $\frac{1}{L!} \sum_{\vec{c} \in \Gamma} \binom{J}{\vec{c}} =$ _____

b Perm $\pi := [6, 7, 8, 1, 2, 3, 4, 5]$ has $\text{Sgn}(\pi) = +\mathbf{1} -\mathbf{1}$.

c Perm $\beta \in \mathbb{S}_{15}$ has sig $[5^3]$. It has _____ many sqroots with sig $[5^3]$, and _____ with sig $[10^1, 5^1]$.

*In grammatical English sentences, TYPE the essay on every **third** line (usually), so that I can easily write between the lines. Do **not** restate the question.*

B2: Let $h(N) := [3^{N-1} - 2^{N-1}] - \frac{1}{2}[3^{N-1} - 1]$. PROVE: Each N in $[3.. \infty)$ yields equality $\mathcal{S}(N, 3) = h(N)$.

Can you give a bijective proof? Does equality hold for $N = 0, 1, 2$?

Can you state and prove a formula for $\mathcal{S}(N, 4)$?

B3: The **4b4** (4 by 4) slider-puzzle has *starting-psn* $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & \bullet \end{bmatrix}$. The lower-right cell is empty. Two

cells are *adjacent* IFF they are adjacent *vertically* or *horizontally*. A *move* is sliding a token adjacent to the emptycell into the emptycell. E.g, sliding **12**↓ yields $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & \bullet \\ 13 & 14 & 15 & 12 \end{bmatrix}$. Now sliding **11**↗ then **15**↑ produces

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 15 & 11 \\ 13 & 14 & \bullet & 12 \end{bmatrix}$. A *config*[uration] is *obtainable* (**Ob**) if it can be obtained by a sequence of moves from the starting-psn.

i Consider an obtainable config Λ . Prove that if two adjacent tokens are interchanged [you pry them out with a screwdriver, switch them, and jam them back in], then the resulting config is *not-obtainable* (**Nob**). [Hint: Encode configs into perms, interpret a *move* as doing something to the perm, then use perm-parity to exhibit an algebraic obstruction to **Ob**.]

ii The above shows that half the configs are **Nob**. Prove that all the others **are** obtainable. I.e, prove that of the $15!$ configs with \bullet in lower-right, that exactly half are **Ob**.

iii Can you generalize to $N \times K$ boards? Can you generalize to non-rectangular boards? E.g, how general a connected finite-subset of the $\mathbb{Z} \times \mathbb{Z}$ has all non-parity-obstructed configs actually **Ob**?

End of Home-B

B1: _____ 55pts

B2: _____ 50pts

B3: _____ 105pts

Total: _____ 210pts

HONOR CODE: “I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague).” *Name/Signature/Ord*

Ord: _____

Ord: _____

Ord: _____

Note: Staple this problem-sheet to your essay. The problem-sheet is the first page of what you hand in.