Staple!

Team B ____

Combinatorics
MAD4203 3214

Home-B

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Due **BoC, Monday, 16Oct2017**, Please *fill-in* every *blank* on this sheet. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

Use $\mathcal{S}(N, K)$ for *Stirling# of 2nd kind.*

A perm of cycle-signature $\lceil 7^3, 4^2, 1^8 \rfloor$ has three 7-cycles, two 4-cycles, and eight fixed-pts. Use ***CN*** for "cycle notation", and ***CCN*** for "canonical CN".

B1: *Henceforth, show no work. Simply fill-in each blank on the problem-sheet.*

___ ___ a For a tuple such as $\vec{\mathbf{c}} := (5, 1, 2)$, let $\binom{8}{\vec{\mathbf{c}}}$ mean multinom-coeff $\binom{8}{5,1,2}$. Fixing natnums J and L, let Γ denote the set of length-L compositions of J. What shorthand do we have from class for this sum? $\frac{1}{L!}\sum_{\vec{\mathbf{c}}\in\Gamma}\binom{J}{\vec{\mathbf{c}}} =$ $\mathcal{S}(J, L)$.

___ ___ b Perm $\pi := [6, 7, 8, 1, 2, 3, 4, 5]$ has $\text{Sgn}(\pi) = +1$ -1.

Perm-sign: This π switches a 3-block with a 5-block, hence can be realized with $3 \cdot 5 = 15$ transpositions, hence is an odd perm.

Alternatively, since $3 \perp 8$, this block-switch makes π a single 8-cycle; the number of even-len-cycles [***EL-cycles***] is 1. Thus $\text{Sgn}(\pi) = [-1]^1 = -1$.

___ ___ c Perm $\beta \in \mathbb{S}_{15}$ has sig $\lceil 5^3 \rfloor$. It has 1 many sqroots with sig $\lceil 5^3 \rfloor$, and $\binom{3}{2,1}\cdot 5^1 = 15$ with sig $\lceil 10^1, 5^1 \rfloor$.

In grammatical English sentences, TYPE *the essay on every third line* (usually), *so that I can easily write between the lines. Do* **not** *restate the question.*

B2: Let $h(N) := [3^{N-1} - 2^{N-1}] - \frac{1}{2}[3^{N-1} - 1]$. PROVE: *Each N in $[3..\infty)$ yields equality $\mathcal{S}(N, 3) = h(N)$.*

Can you give a bijective proof? Does equality hold for $N = 0, 1, 2$?

Can you state and prove a formula for $\mathcal{S}(N, 4)$?

B3: The ***4b4*** (4 by 4) slider-puzzle has ***starting-psn*** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & \bullet \end{bmatrix}$. The lower-right cell is empty. Two cells are ***adjacent*** IFF they are adjacent *vertically* or *horizontally*. A ***move*** is sliding a token adjacent to the emptycell into the emptycell. E.g, sliding $12\downarrow$ yields $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & \bullet \\ 13 & 14 & 15 & 12 \end{bmatrix}$. Now sliding $\overrightarrow{11}$ then $15\uparrow$ produces $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 15 & 11 \\ 13 & 14 & \bullet & 12 \end{bmatrix}$. A ***config***[uration] is ***obtainable*** (***Ob***) if it can be obtained by a seqence of moves from the starting-psn.

i Consider an obtainable config Λ. Prove that if two adjacent tokens are interchanged [you pry them out with a screwdriver, switch them, and jam them back in], then the resulting config is *not-obtainable* (***Nob***). [*Hint:* Encode configs into perms, interpret a *move* as doing something to the perm, then use perm-parity to exhibit an algebraic obstruction to **Ob**.]

ii The above shows that half the configs are **Nob**. Prove that all the others **are** obtainable. I.e, prove that of the 15! configs with $\bullet$ in lower-right, that exactly half are **Ob**.

iii Can you generalize to $N\times K$ boards? Can you generalize to non-rectangular boards? E.g, how general a connected finite-subset of the $\mathbb{Z}\times\mathbb{Z}$ has all non-parity-obstructed configs actually **Ob**?

End of Home-B

B1:	___ ___	55pts
B2:	___ ___	50pts
B3:	___ ___ ___	105pts
Total:	___ ___ ___	210pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* Name/Signature/Ord

Ord: ....

Ord: ....

Ord: ....

Note: Staple this problem-sheet to your essay. The problem-sheet is the first page of what you hand in.