Advanced Calc
MAA4102 & MAA5104

Prac-B

Prof. JLF King
17Jul2018

Please. Use "$f(x)$ notation" when writing fncs; in particular, for trig and log fncs. E.g, write "$\sin(x)$" rather than the horrible $\sin x$ or $[\sin x]$. Write expressions unambiguously e.g, "$1/a+b$" should be bracketed either $[1/a]+b$ or $1/[a+b]$. (Be careful with **negative** signs!)

Abbrevs: **OYOSOPaper** for "on your own sheets of paper"; you are writing a math essay, giving a definition or a proof. No "scratch work" accepted, only complete, grammatical, coherent sentences. Write **every 2nd or every 3rd line** for math essays.

ITOf for "in terms of". **st.** for "such that" **seq** for "sequence" **posint** for "positive integer" **DL** for "Dedekind-Left", the lefthand atom of a Dedekind cut. **poly** for "polynomial" **coeff** for "coefficient"

For each of the limit questions, write "+∞", "−∞", a real number, or –*if none of these*– "DNE". Do ***not*** approx.: If your result is "$\sin(\sqrt{\pi})$" then write that rather than $.9797\cdots$. Write **DNE** in a blank <u>if</u> the described object does not exist or if the indicated operation cannot be performed.

B0: For the following Greek letters, please write the **other case**, along with their names.

Eg: "α: ______ B: ______." You fill in: <u>A *alpha*</u> <u>β *beta*</u>.

Ω: ______ Γ: ______ Ψ: ______

μ: ______ ρ: ______ δ: ______

B1: Show no work.

a) $\lim_{n\to\infty} \frac{5n+[2+3n][n-1]}{4+n^2} =$ ___.

b) (Right/left hand limits): $\lim_{x\to 3^+} \frac{|3-x|}{3-x} =$ ______.

$\lim_{x\to 3^-} \frac{|3-x|}{3-x} =$ ______.

c) The set $S\subset\mathbb{R}$ is upper-bounded, yet has no LUB (in $\mathbb{R}$). So $S=$ ______.

A particular upper-bnd for *your* S is ______ $\in\mathbb{R}$.

d) For $r\in[-\infty,+\infty]$, its DL is $\{q\in\mathbb{Q} \mid q<r\}$, written $\mathbf{L}_r$.

Given $s,t\in\mathbb{R}$, define "$s+t$" by defining

$\mathbf{L}_{s+t} :=$ ______

ITOf $\mathbf{L}_s$ and $\mathbf{L}_t$ and addition-of-rationals.

If you apply this defn to $s:=-\infty$ and $t:=+\infty$, the definition says that $-\infty+[+\infty]=$ ______.

e) For a set S of extended reals, define its supremum $r:=\sup(S)$ by defining its DL:

$\mathbf{L}_r :=$ ______.

f) Use $\lfloor\cdot\rfloor$ for the "floor fnc" (the "greatest integer fnc"). So $\lfloor\boldsymbol{\pi}\rfloor=$ ______ and $\lfloor-\boldsymbol{\pi}\rfloor=$ ______.

g) Let E be the set $\left\{5+\left[[-1]^n\cdot\frac{3n-1}{n}\right]\right\}_{n\in\mathbb{Z}_+}$. Then $\sup_{\mathbb{R}}(E)=$ ______ and $\inf_{\mathbb{R}}(E)=$ ______.

h) $\lim_{x\to 0} \frac{\sqrt{9+x}-\sqrt{9-x}}{x} =$ ______.

i) Let $B(x):=x^{[2x]}$. Its derivative, then, is

$B'(x)=$ ______.

[*Hint:* How is $x^{[2x]}$ defined ITOf the exponential fnc?]

Essays. On your own sheets of lined paper, give the following definitions or proofs. No "scratch work" accepted, only complete, grammatical, coherent sentences. Write **every 2nd or every 3rd line** for math essays.

B2.14159: The ***P.F*** [Prime Factorization] of a posint k is the unique finite multiset $\mathcal{S}_k\subset$Primes st. $\prod(\mathcal{S}_k)=k$. E.g $\mathcal{S}_{90}=\{2,3,3,5\}$.

Let $\equiv$ mean $\equiv_3$. A **T-*integer*** (abbrev, ***tint***) is an element of

$$\mathbf{T} := \{n\in\mathbb{Z}_+ \mid n\equiv 1\} \overset{\text{note}}{=} \{1,4,7,10,13,\dots\}.$$

A tint α is ***amber*** if $\alpha\neq 1$ and

$$\forall x\in\mathbf{T}:\quad x \bullet\!| \ \alpha \implies x\in\{1,\alpha\}.$$

Tint β is ***blue*** if $\beta\neq 1$ and

$$\forall y,z\in\mathbf{T}:\quad \beta\bullet\!|\ [y\cdot z] \implies \left[[\beta\bullet\!|\ y] \text{ or } [\beta\bullet\!|\ z]\right].$$

(E.g: 10 is amber, but certainly not blue, since $10\bullet\!|\ [4\cdot 25]$, yet 10 divides neither 4 nor 25).

i State and prove: THM A. *A tint $\alpha>1$ is amber IFF its P.F $\mathcal{S}_\alpha$ satisfies...*

For each tint below, either write "*amber*" or else a **T**-factorization $w\cdot x$, with $w,x \in \mathbf{T}\setminus\{1\}$.

49: .... 52: .... 55: .... 58: ....

ii State and prove: THM B. *Tint $\beta>1$ is blue IFF its P.F $\mathcal{S}_\beta$ fulfills...*

In each blank, write "*blue*" or $y\cdot z$ with $y,z \in \mathbf{T}$ yet $\beta \nmid y$ and $\beta \nmid z$.

55• .... 79• .... 121• ....

B2: Recall $\boldsymbol{\mathcal{B}} := 1 + 3\mathbb{N}$, the set of "Blip-numbers", which is sealed under multiplication. OYOSOPaper, give the formal definitions of:

*A number $\alpha \in \boldsymbol{\mathcal{B}}$ is **$\boldsymbol{\mathcal{B}}$-irreducible** IFF...*

*A Blip number β is **$\boldsymbol{\mathcal{B}}$-prime** IFF...*

Produce two different factorizations of 484 into a product of $\boldsymbol{\mathcal{B}}$-irreds. Produce, with careful proof, a Blip-number which is $\boldsymbol{\mathcal{B}}$-irreducible but *not* $\boldsymbol{\mathcal{B}}$-prime. [*Hint:* Make use of your factorizations of 484.]

B3: i Write the *definition* of "$\lim_{x\to 4} x^2 = 5$". [*Hint:* You need 3 quantifiers, an ε and a δ.]

ii Given seqs. with $\lim(\vec{\mathbf{b}}) = 2$ and $\lim(\vec{\mathbf{c}}) = 4$, let $\vec{\mathbf{a}} := \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$. Given $\varepsilon>0$, produce –with careful proof– a posint N st.: $\forall k \geqslant N\colon\ |a_k - 8| < \varepsilon$.

iii Given seqs. with $\lim(\vec{\mathbf{b}}) = 2$ and $\lim(\vec{\mathbf{c}}) = 4$, let $\vec{\mathbf{s}} := \vec{\mathbf{b}} + \vec{\mathbf{c}}$. Given $\varepsilon>0$, produce –with careful proof– a posint N st.: $\forall k \geqslant N\colon\ |s_k - 6| < \varepsilon$.

B4: Consider a seq. $\vec{\mathbf{b}}$ of distinct reals. Define the index-set

$$\mathbf{T} := \{N \in \mathbb{Z}_+ \mid \forall k > N\colon\ b_N > b_k\}.$$

Use $\mathbf{T}$ to give a careful proof of: *Either $\vec{\mathbf{b}}$ has an increasing subseq, or a decreasing subsequence.*

B5: State the Triangle Inequality (TE). Start your defn with "The TE states that for all reals...".

State Bernoulli's Inequality (BE). Now give a careful induction proof of BE.

B6: State the Fundamental Thm of Arithmetic (about factoring into primes).

Use the FTArithmetic to give a careful proof that there are *no* posints n, d for which $\left[\frac{n}{d}\right]^2 = 7$.

B7: State the Sandwich Theorem. [*Hint:* It relates the limits of three sequences.] Now prove the Sandwich Theorem.

B8: Define: "A totally-ordered set $(\mathbb{T}, <)$ has the ***LUB-property*** IFF..."

Give a careful proof of: *If $\mathbb{T}$ has the LUB-property, then $\mathbb{T}$ has the GLB-property.*

B9: Define: "On a set E, a binary relation ∇ is an ***equivalence relation*** IFF...". *Make sure* to define any terms like "reflexive" that you use in your defn.!

Let $\mathbf{P}$ be the set of ordered integer-pairs (n, d), with $d \neq 0$. Define relation \$ on $\mathbf{P}$ by

$$(N, D)\ \$\ (x, y) \qquad \text{IFF} \qquad N \cdot y = x \cdot D.$$

Prove, in detail, that \$ is an equivalence relation.

B10: Describe, in detail, the (necessarily erroneous) "proof" that all horses are the same color. Precisely write the meaning of the N^{th} proposition.

Now, examining the quantifiers carefully, pinpoint the exact error in the "proof". (Warning: I'm not talking about an error in the result; pinpoint the error in *the reasoning*).

B11: Define: "A number $\beta \in \mathbb{R}$ is "***algebraic*** of degree-3" IFF...". You may use "poly(nomial)" without defn, but if you use terms "ratpoly" or "intpoly" then you must define them. Note that ***zip*** is my name for the poly all of whose coeffs are zero.

Define: "A number $\tau \in \mathbb{R}$ is ***transcendental*** IFF..."

End of Prac-B

Print name .... Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ....