Advanced Calc
MAA4102 & MAA5104

# Home-$\alpha$

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**Definitions.** Say that two integers $p$ and $q$ (not both zero) are ***relatively prime***, written $p \perp q$, if $\operatorname{Gcd}\{p, q\}$ is 1. A rational $\frac{p}{q}$ is in ***LCTerms*** (lowest common terms) if $p \perp q$.

A rational number $x$ is a ***dyadic rational*** if, in LCTerms, it is $p/2^n$, with $p \in \mathbb{Z}$ and $n$ a natnum. Thus each integer $k$ is dyadic, since $k = k/2^0$. Also $\frac{-17}{32}$ and $\frac{5}{10}$ are dyadic rationals, but $\frac{1}{10}$ is not, and neither is $\pi$ nor $\sqrt{32}$. Use ***cty*** and ***discty*** to abbreviate "continuity" and "discontinuity".

Suppose that $B$ is a set of reals (such as an interval) and we have a function $f\colon B \to \mathbb{R}$. Let $\operatorname{Cty}(f)$ denote the set of $x \in B$ at which $f$ is continuous. Let $\operatorname{DisCty}(f)$ denote the set of $x \in B$ at which $f$ is *dis*continuous.

For a set $S \subset \mathbb{R}$, let $\mathbf{1}_S$ denote the **"*indicator function* of $S$"**: For $x \in \mathbb{R} \smallsetminus S$, then, $\mathbf{1}_S(x) := 0$. And $\mathbf{1}_S(x) := 1$, for $x$ in $S$.

**Advice.** Carefully follow the writing style described on *The Checklist*. Print a copy of your essay(s) each day, so that you always have a paper copy. Each team member is responsible to understand everything that the team hands in.

**$\alpha$1:** Fix two arbitrary sequences $\mathbf{a}$ and $\mathbf{b}$. Define

$$P(N) := a_N b_0 + a_{N-1} b_1 + a_{N-2} b_2 + \cdots + a_0 b_N$$

and

$$\begin{aligned} Q(N) := \quad & a_0 \cdot [b_0 + \cdots + b_{N-1} + b_N] \\ &+ [a_1 - a_0] \cdot [b_0 + \cdots + b_{N-1}] \\ &+ [a_2 - a_1] \cdot [b_0 + \cdots + b_{N-2}] \\ &+ [a_3 - a_2] \cdot [b_0 + \cdots + b_{N-3}] \\ &\quad \vdots \\ &+ [a_{N-1} - a_{N-2}] \cdot [b_0 + b_1] \\ &+ [a_N - a_{N-1}] \cdot b_0 \,. \end{aligned}$$

Use induction on $N$ to prove the summation formula $P(N) = Q(N)$, for $N = 0, 1, 2, \ldots$.

**$\alpha$2:** Show no work for this problem. (No partial credit.) Each function below is $\mathbb{R} \to \mathbb{R}$.

a) Let $S$ be the set of reals $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$. Then

$\operatorname{DisCty}(\mathbf{1}_S) =$ ________________.

b) $\operatorname{Cty}(\mathbf{1}_{\mathbb{Q}}) =$ ________________.

**$\alpha$3:** Let $\mathbb{D} \subset \mathbb{Q}$ be the set of *dyadic rationals*. Define the ***ruler function*** $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}$ by:

$$\mathbf{r}(x) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{D} \text{ and } x = \frac{p}{q} \text{ in LCTerms;} \\ 0 & \text{if } x \in \mathbb{R} \smallsetminus \mathbb{D}. \end{cases}$$

Thus $\mathbf{r}(0) = 1$ and $\mathbf{r}(\frac{-6}{64}) = \frac{1}{32}$ and $\mathbf{r}(\frac{1}{3}) = 0$ and $\mathbf{r}(\boldsymbol{\pi}) = 0$.

Prove that $\mathbf{r}$ is discontinuous at $x$ IFF $x$ is a dyadic rational. In particular, for each dyadic $P$, give a **"*witness* of discty of $\mathbf{r}()$ at $P$"**. I.e, give a *particular* positive number $\varepsilon := \operatorname{Formula}(P)$ and sequence $\mathbf{x} := \operatorname{Formula}(P)$, such that

$$\forall n\colon \quad \big|\mathbf{r}(x_n) - \mathbf{r}(P)\big| \geqslant \varepsilon\,,$$

yet $\lim(\mathbf{x})$ equals $P$.

**$\alpha$4:** Give an example of a sequence $\mathbf{b}$ of **integers**, so that for each positive integer $L$, there is a list $(n_k)_{k=1}^{\infty}$ so that $\lim_{k\to\infty} b_{n_k}$ equals $L$.

**$\alpha$5:** *Create* an interesting AdvCalc problem, then carefully solve it. (I may put such a problem on the next exam.) Also acceptable: Pick a new interesting problem from the text, and solve it.

End of Home-$\alpha$