Miscellaneous Algebra facts

23 September, 2017 (at *14:35*)

Ways to count in groups

1: Lagrange's Theorem. *Given groups $H \subset G$, then,* $\mathrm{Ord}(H) \bullet\!\mid \mathrm{Ord}(G)$. ◊

Proof. The left-cosets of H form a partition of G. ♦

The symbol $G \circlearrowleft \Omega$ means that gp G ***acts on*** set Ω; there is a gp-hom $\boxed{\psi{:}G\to\mathbb{S}_\Omega}$. For $g \in G$ and $\omega \in \Omega$, write the gp-action as $\psi_g(\omega)$ or $g(\omega)$ or just $g\omega$. Define the ***orbit*** and ***stabilizer*** of a point ω, and the ***fixed-pt set*** of a group-element g:

$$\begin{aligned} \mathcal{O}_\psi(\omega) &\coloneqq \{g\omega \mid g \in G\} && \subset \Omega\,; \\ \mathrm{Stab}_\psi(\omega) &\coloneqq \{g \in G \mid g\omega = \omega\} && \subset G\,; \\ \mathrm{Fix}_\psi(g) &\coloneqq \{\omega \in \Omega \mid g\omega = \omega\} && \subset \Omega\,. \end{aligned}$$

This $\mathrm{Stab}(\omega)$ is a subgp, but is rarely normal in G:

2: $$\forall f \in G\colon\ f \cdot \mathrm{Stab}(\omega) \cdot f^{-1} = \mathrm{Stab}(f\omega)\,.$$

3: Orb-Stab Lemma. *For each $\omega \in \Omega$:*

$*$: $$\mathrm{Ord}\big(\mathrm{Stab}(\omega)\big) \cdot \big|\mathcal{O}(\omega)\big| = \mathrm{Ord}\big(G\big)\,.$$ ◊

Proof. Let $H \coloneqq \mathrm{Stab}(\omega)$. Say two elts $g, f \in G$ are "equivalent", $g \sim f$, if $g\omega = f\omega$. Evidently, the equiv-class of g is simply the left coset gH. These equiv-classes partition G, hence $(*)$. ♦

4: Burnside's Lemma. *Counting cardinalities,*

†: $$\sum_{\omega\in\Omega} |\mathrm{Stab}(\omega)| \overset{\#}{=} \Big\{ (g,\omega) \;\Big|\; g\omega = \omega \Big\} \overset{\#}{=} \sum_{g\in G} |\mathrm{Fix}(g)|\,.$$

Counting the number of G-orbits, then,

‡: $$\begin{aligned} {}^{\#}\mathit{Orbits} &= \frac{1}{|G|}\cdot\sum_{g\in G}\Big|\mathrm{Fix}(g)\Big| \\ &= \left[\text{\# of points fixed by an average element of } G\right]. \end{aligned}$$ ◊

Proof. The number of G-orbits equals

$$\sum_{\omega\in\Omega} \frac{1}{|\mathcal{O}(\omega)|} \overset{\text{Orb-Stab, }(3*)}{=\!=\!=} \frac{1}{|G|}\cdot\sum_{\omega\in\Omega} |\mathrm{Stab}(\omega)|\,.$$

Now apply (4†) to earn (4‡). ♦

Applications of counting. Color the 6-faces of a cube red, white and blue. How many distinct colorings are there, up to orientation preserving rotation? We will use Burnside's Lemma.

What isometry g?	How many such g?	${}^{\#}\mathrm{Fix}(g) = 3^F$	$F := {}^{\#}$[Face-orbits under g]
Id	1	3^6	1+1+1+1+1+1
FaceRot 90°	$\frac{6}{2}\cdot 2 = 6$	3^3	1+4+1
FaceRot 180°	$\frac{6}{2}\cdot 1 = 3$	3^4	1+2+2+1
VertexRot 120°	$\frac{8}{2}\cdot 2 = 8$	3^2	3+3
EdgeRot 180°	$\frac{12}{2}\cdot 1 = 6$	3^3	2+2+2

The sum $\frac{1}{24}\cdot[1\cdot 3^6 + 6\cdot 3^3 + 3\cdot 3^4 + 8\cdot 3^2 + 6\cdot 3^3]$ equals 57. Using K many colors, the number of K-colorings is $\frac{1}{24}\cdot[K^6 + 3K^4 + 12K^3 + 8K^2]$, i.e, is

$$K^2\cdot[K^4 + 3K^2 + 12K + 8]/24\,.$$

Filename: `Problems/Algebra/alg.misc.latex`
As of: *Tuesday 07Mar2006.* Typeset: *23Sep2017* at *14:35.*