
Ways to count in groups

1: Lagrange's Theorem. Given groups $H \subset G$, then, $\text{Ord}(H) \bullet \mid \text{Ord}(G)$. \blacklozenge

Proof. The left-cosets of H form a partition of G . \blacklozenge

The symbol $G \curvearrowright \Omega$ means that gp G **acts on** set Ω ; there is a gp-hom $\boxed{\psi: G \rightarrow \mathbb{S}_\Omega}$. For $g \in G$ and $\omega \in \Omega$, write the gp-action as $\psi_g(\omega)$ or $g(\omega)$ or just $g\omega$. Define the **orbit** and **stabilizer** of a point ω , and the **fixed-pt set** of a group-element g :

$$\begin{aligned} \mathcal{O}_\psi(\omega) &:= \{g\omega \mid g \in G\} && \subset \Omega; \\ \text{Stab}_\psi(\omega) &:= \{g \in G \mid g\omega = \omega\} && \subset G; \\ \text{Fix}_\psi(g) &:= \{\omega \in \Omega \mid g\omega = \omega\} && \subset \Omega. \end{aligned}$$

This $\text{Stab}(\omega)$ is a subgp, but is rarely normal in G :

2: $\forall f \in G: f \cdot \text{Stab}(\omega) \cdot f^{-1} = \text{Stab}(f\omega).$

3: Orb-Stab Lemma. For each $\omega \in \Omega$:

*: $\text{Ord}(\text{Stab}(\omega)) \cdot |\mathcal{O}(\omega)| = \text{Ord}(G).$

Proof. Let $H := \text{Stab}(\omega)$. Say two elts $g, f \in G$ are “equivalent”, $g \sim f$, if $g\omega = f\omega$. Evidently, the equiv-class of g is simply the left coset gH . These equiv-classes partition G , hence (*). \blacklozenge

4: Burnside's Lemma. *Counting cardinalities,*

$$\dagger: \sum_{\omega \in \Omega} |\text{Stab}(\omega)| \stackrel{\#}{=} \left\{ (g, \omega) \mid g\omega = \omega \right\} \stackrel{\#}{=} \sum_{g \in G} |\text{Fix}(g)|.$$

Counting the number of G -orbits, then,

$$\ddagger: \begin{aligned} \# \text{Orbits} &= \frac{1}{|G|} \cdot \sum_{g \in G} |\text{Fix}(g)| \\ &= \left[\begin{array}{l} \# \text{ of points fixed by an av-} \\ \text{erage element of } G \end{array} \right]. \end{aligned}$$

Proof. The number of G -orbits equals

$$\sum_{\omega \in \Omega} \frac{1}{|\mathcal{O}(\omega)|} \stackrel{\text{Orb-Stab, (??*)}}{=} \frac{1}{|G|} \cdot \sum_{\omega \in \Omega} |\text{Stab}(\omega)|.$$

Now apply (??†) to earn (??‡). ♦

Applications of counting. Color the 6-faces of a cube red, white and blue. How many distinct colorings are there, up to orientation preserving rotation? We will use **Burnside's Lemma**.

What isometry g ?	How many such g ?	$\# \text{Fix}(g) = 3^F$	$F := \# [\text{Face-orbits under } g]$
Id	1	3^6	1+1+1+1+1+1
FaceRot 90°	$\frac{6}{2} \cdot 2 = 6$	3^3	1+4+1
FaceRot 180°	$\frac{6}{2} \cdot 1 = 3$	3^4	1+2+2+1
VertexRot 120°	$\frac{8}{2} \cdot 2 = 8$	3^2	3+3
EdgeRot 180°	$\frac{12}{2} \cdot 1 = 6$	3^3	2+2+2

The sum $\frac{1}{24} \cdot [1 \cdot 3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 8 \cdot 3^2 + 6 \cdot 3^3]$ equals 57. Using K many colors, the number of K -colorings is $\frac{1}{24} \cdot [K^6 + 3K^4 + 12K^3 + 8K^2]$, i.e, is

$$K^2 \cdot [K^4 + 3K^2 + 12K + 8]/24.$$