

Affine maps of the plane

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Entrance. A matrix $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts on a point $\begin{bmatrix} x \\ y \end{bmatrix}$, sending it to $M\begin{bmatrix} x \\ y \end{bmatrix}$. Let $\hat{\mathbf{i}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{\mathbf{j}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let

$$R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

be the std rotation matrix.

Recall that SL_2 , the *special linear group*, is the group (sealed under matrix-mult and matrix inverse) of 2×2 matrices M with $\text{Det}(M) = 1$. Each $M \in SL_2$ is **OPAP**: *Orientation Preserving*, since $\text{Det}(M) > 0$; and *Area Preserving*, since $|\text{Det}(M)| = 1$.

Shears. A real m yields horizontal/vertical shears:

$$H_m := \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad V_m := \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}.$$

Abbrev. “horizontal(ly) shear” by *hshear*, and “vertical(ly) shear” by *vshear*. Call a horizontal-or-vertical shear a “perp-shear”. Let Σ denote the group generated by perp-shears.

1: Perp-shear Lemma. *Rotation R_π is a product of perp-shears: $H_{-2}V_1H_{-2}V_1$.*

a: For each angle θ , rotation R_θ is a product of at most 5 perp-shears.

b: The group generated by the perp-shears is precisely SL_2 . \diamond

Pf of (a). WLOG $\theta \in (0, \pi)$. Let $\mathbf{q} = \begin{bmatrix} r \\ s \end{bmatrix}$ be the unit-vector at angle θ . Vertically shear $\hat{\mathbf{i}}$ up to height s , then over to be \mathbf{q} , i.e take $\alpha \in \mathbb{R}$ st. $[H_\alpha V_s]\hat{\mathbf{i}} = \mathbf{q}$. This action moves $\hat{\mathbf{j}}$ to some vector $\mathbf{w} := [H_\alpha V_s]\hat{\mathbf{j}}$.

Let \mathbb{L} be the line parallel to vector \mathbf{q} , and passing through point $R_{\pi/2}(\mathbf{q})$. Since shears are OPAP, this \mathbf{w} must lie on \mathbb{L} .

Take the $z \in \mathbb{R}$ which vshears \mathbf{q} onto the horiz-axis, i.e $V_z\mathbf{q} = \begin{bmatrix} r \\ 0 \end{bmatrix}$. For each $\beta \in \mathbb{R}$, then, the hshear H_β fixes $V_z\mathbf{q}$. Since $V_z\mathbf{q}$ is on the horiz-axis, point $V_z\mathbf{w}$ cannot be [they are two edges of an area=1 parallelogram]. Hence $\{[H_\beta V_z]\mathbf{w} \mid \beta \in \mathbb{R}\}$ is an entire line (horizontal, since vector $V_z\mathbf{q}$ is horizontal). Letting $T_\beta := V_{-z}H_\beta V_z$, then,

$\{[T_\beta]\mathbf{w} \mid \beta \in \mathbb{R}\}$ is all of \mathbb{L} .

Thus there is a particular β -value, b , st. $[T_b]\mathbf{w}$ is orthogonal to $[T_b]\mathbf{q} \stackrel{\text{note}}{=} \mathbf{q}$. Since \mathbf{q} has length 1, our $[T_b]\mathbf{w}$ must have length 1. Thus $[T_b]\mathbf{w}$ is $[T_b]\mathbf{q}$ hit by $R_{\pi/2}$. I.e,

$$\begin{aligned} [T_b]\mathbf{q} &= [R_\theta]\hat{\mathbf{i}}, \quad \text{and} \\ [T_b]\mathbf{w} &= [R_\theta]\hat{\mathbf{j}}. \end{aligned}$$

Consequently

$$\begin{aligned} [R_\theta]\hat{\mathbf{i}} &= [T_b]\mathbf{q} = [T_b H_\alpha V_s]\hat{\mathbf{i}}, \quad \text{and} \\ [R_\theta]\hat{\mathbf{j}} &= [T_b]\mathbf{w} = [T_b H_\alpha V_s]\hat{\mathbf{j}}. \end{aligned}$$

Thus R_θ equals $T_b H_\alpha V_s$, a product of 5 perp-shears. \blacklozenge

Proof of (b). To show that a $T \in SL_2$ is a perp-shear product, let $\mathbf{u} := T\hat{\mathbf{i}}$ and $\mathbf{v} := T\hat{\mathbf{j}}$. We'll carry pair (\mathbf{u}, \mathbf{v}) to $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ via perp-shears.

Take a rotation R st. $\begin{bmatrix} r \\ s \end{bmatrix} := R\mathbf{u}$ has $0 < s < 1$. So there is an hshear H st. $HR\mathbf{u}$ has length 1. Now take the rotation R' st. $R'HR\mathbf{u}$ is $\hat{\mathbf{i}}$.

All this has carried \mathbf{v} to $\mathbf{v}' := [R'HR]\mathbf{v}$. Since $(\hat{\mathbf{i}}, \mathbf{v}')$ defines a parallelogram with signed-area 1, there is a (unique) hshear H' st. $H'\mathbf{v}' = \hat{\mathbf{j}}$. The upshot: $H'R'HR$ carries pair (\mathbf{u}, \mathbf{v}) to $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$. So $T := [H'R'HR]^{-1}$. \blacklozenge

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