Affine maps of the plane

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Entrance. A matrix $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts on a point $[z \ y]_G$, sending it to $M[z \ y]_G$. Let $\hat{i} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let

$$R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

be the std rotation matrix.

Recall that $SL_2$, the special linear group, is the group (sealed under matrix-mult and matrix inverse) of $2\times2$ matrices $M$ with $\text{Det}(M) = 1$. Each $M \in SL_2$ is OPAP: Orientation Preserving, since $\text{Det}(M) > 0$; and Area Preserving, since $|\text{Det}(M)| = 1$.

Shears. A real $m$ yields horizontal/vertical shears:

$$H_m := \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \quad V_m := \begin{bmatrix} 1 \\ m \end{bmatrix}.$$ 

Abbrev. “horizontal(ly) shear” by $\text{hshear}$, and “vertical(ly) shear” by $\text{vshear}$. Call a horizontal-or-vertical shear a “perp-shear”. Let $\Sigma$ denote the group generated by perp-shears.

1: Perp-shear Lemma. Rotation $R_\pi$ is a product of perp-shears: $H_2V_1H_2V_1$.

a: For each angle $\theta$, rotation $R_\theta$ is a product of at most 5 perp-shears.

b: The group generated by the perp-shears is precisely $SL_2$.

\textbf{Pf of (a).} WLOG $\theta \in (0, \pi)$. Let $q = \begin{bmatrix} s \\ 0 \end{bmatrix}$ be the unit-vector at angle $\theta$. Vertically shear $\hat{i}$ up to height $s$, then over to be $q$, i.e take $\alpha \in \mathbb{R}$ st. $[H_\alpha V_s] \hat{i} = q$. This action moves $\hat{j}$ to some vector $w := [H_\alpha V_s] \hat{j}$. Let $L$ be the line parallel to vector $q$, and passing through point $R_{\pi/2}(q)$. Since shears are OPAP, this $w$ must lie on $L$.

Take the $z \in \mathbb{R}$ which vshears $q$ onto the horizon-axis, i.e $V_z q = \begin{bmatrix} 0 \\ z \end{bmatrix}$. For each $\beta \in \mathbb{R}$, then, the hshear $H_\beta$ [fixes $V_z q$]. Since $V_z q$ is on the horizon-axis, point $V_z w$ cannot be [they are two edges of an area=1 parallelogram]. Hence $\{[H_\beta V_z] w \mid \beta \in \mathbb{R}\}$ is an entire line (horizontal, since vector $V_z q$ is horizontal). Letting $T_\beta := V_z H_\beta V_z$, then,

$$\{[T_\beta] w \mid \beta \in \mathbb{R}\} \text{ is all of } L.$$ 

Thus there is a particular $\beta$-value, $b$, st. $[T_b] w$ is orthogonal to $[T_b] q \not\equiv q$. Since $q$ has length 1, our $[T_b] w$ must have length 1. Thus $[T_b] w$ is $[T_b] q$ hit by $R_\pi/2$. I.e,

$$[T_b] q = [R_\theta] \hat{i}, \quad \text{and} \quad [T_b] w = [R_\theta] \hat{j}.$$ 

Consequently

$$[R_\theta] \hat{i} = [T_b] q = [T_b H_\alpha V_s] \hat{i}, \quad \text{and} \quad [R_\theta] \hat{j} = [T_b] w = [T_b H_\alpha V_s] \hat{j}.$$ 

Thus $R_\theta$ equals $T_b H_\alpha V_s$, a product of 5 perp-shears.  

\textbf{Proof of (b).} To show that a $T \in SL_2$ is a perp-shear product, let $u := T \hat{i}$ and $v := T \hat{j}$. We’ll carry pair $(u, v)$ to $(\hat{i}, \hat{j})$ via perp-shears.

Take a rotation $R$ st. $[s] := Ru$ has $0 < s < 1$. So there is an $\text{hshear}$ $H$ st. $HRu$ has length 1. Now take the rotation $R'$ st. $R' HRu$ is $\hat{i}$.

All this has carried $v$ to $v' := R' HRv$. Since $(\hat{i}, v')$ defines a parallelogram with signed-area 1, there is a (unique) hshear $H'$ st. $H' v' = \hat{j}$. The upshot: $H' R' HR$ carries pair $(u, v)$ to $(\hat{i}, \hat{j})$. So $T := [H' R' HR]^{-1}$.  

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