Multiplication in $\mathbb{C}$ is continuous

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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Abbreviations. Use ***posreal*** for "positive real number". A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_1, x_2, x_3, \ldots)$. Use $\text{Tail}_N(\vec{\mathbf{x}})$ for the subsequence $(x_N,\ x_{N+1},\ x_{N+2}, \ldots)$ of $\vec{\mathbf{x}}$. □

1: Addition-Cts thm. *The addition operation $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is continuous.* Restated: *Suppose $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{C}$ with $\lim(\vec{\mathbf{x}}) = \alpha$ and $\lim(\vec{\mathbf{y}}) = \beta$. With $p_n := x_n + y_n$, then, $\lim(\vec{\mathbf{p}}) = \alpha + \beta$.* ◊

Proof. Fix a posreal ε. Take N large enough that

$$\text{Tail}_N(\vec{\mathbf{x}}) \subset \text{Bal}_{\frac{\varepsilon}{2}}(\alpha) \quad \text{and} \quad \text{Tail}_N(\vec{\mathbf{y}}) \subset \text{Bal}_{\frac{\varepsilon}{2}}(\beta)\,.$$

Each index k has $p_k - [\alpha+\beta] = [x_k - \alpha] + [y_k - \beta]$. For each $k \geqslant N$, then,

$$\left|p_k - [\alpha+\beta]\right| \leqslant |x_k - \alpha| + |y_k - \beta| \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,. \quad ♦$$

Remark. The same thm and proof hold for addition on a normed vectorspace; simply replace $|\cdot|$ by the norm $\|\cdot\|$. □

Abbreviations. Use ***WELOG*** for "without essential loss of generality", and ***posint*** for "positive integer".

A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_1, x_2, x_3, \ldots)$. Use $\text{Diam}(\vec{\mathbf{x}})$ for the *diameter* of the set $\{x_n\}_{n=1}^{\infty}$. □

2: Mult-Cts thm. *The multiplication operation $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is continuous.* Restated: *Suppose $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{C}$ with $\lim(\vec{\mathbf{x}}) = \alpha$ and $\lim(\vec{\mathbf{y}}) = \beta$. With $p_n := x_n \cdot y_n$, then, $\lim(\vec{\mathbf{p}}) = \alpha \cdot \beta$.* ◊

Proof. WELOG $|\beta| \leqslant 7$. Since $\vec{\mathbf{x}}$ converges, necessarily the $\text{Diam}(\vec{\mathbf{x}})$ is finite; WELOG

$$†: \qquad \forall \text{ posints } n\text{:} \quad |x_n| \leqslant 50\,.$$

For each posint n, adding and subtracting a term gives

$$\begin{aligned} x_n y_n - \alpha\beta &= x_n y_n - x_n\beta \;+\; x_n\beta - \alpha\beta \\ &= x_n[y_n - \beta] \;+\; [x_n - \alpha]\beta\,. \end{aligned}$$

Taking absolute-values, then upper-bounding, yields

$$‡: \quad \begin{aligned} |x_n y_n - \alpha\beta| &\leqslant |x_n|\cdot|y_n - \beta| \;+\; |x_n - \alpha|\cdot|\beta| \\ &\leqslant 50\cdot|y_n - \beta| \;+\; |x_n - \alpha|\cdot 7\,, \end{aligned}$$

by (†) and the first sentence.

Fix a posreal ε. Since $\lim(\vec{\mathbf{y}}) = \beta$ and $\lim(\vec{\mathbf{x}}) = \alpha$, we can take K large enough that for each n in $[K\,..\,\infty)$:

$$|y_n - \beta| \leqslant \frac{\varepsilon/2}{50} \quad \textit{and} \quad |x_n - \alpha| \leqslant \frac{\varepsilon/2}{7}\,.$$

Plugging these estimates in to (‡) gives that

$$|x_n y_n - \alpha\beta| \leqslant 50\cdot\tfrac{\varepsilon/2}{50} + \tfrac{\varepsilon/2}{7}\cdot 7 \stackrel{\text{note}}{=\!=} \varepsilon\,,$$

for *each* $n \geqslant K$.

As this holds for every ε positive, $\lim(\vec{\mathbf{x}}\cdot\vec{\mathbf{y}})$ indeed equals $\alpha\beta$. ♦

Filename: `Problems/Analysis/Calculus/add-mult-are-cts.latex`
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