

Multiplication in \mathbb{C} is continuous

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Abbreviations. Use *posreal* for “positive real number”. A sequence \vec{x} abbreviates (x_1, x_2, x_3, \dots) . Use $\text{Tail}_N(\vec{x})$ for the subsequence $(x_N, x_{N+1}, x_{N+2}, \dots)$ of \vec{x} .

1: Addition-Cts thm. *The addition operation $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is continuous. Restated: Suppose $\vec{x}, \vec{y} \subset \mathbb{C}$ with $\lim(\vec{x}) = \alpha$ and $\lim(\vec{y}) = \beta$. With $p_n := x_n + y_n$, then, $\lim(\vec{p}) = \alpha + \beta$.* ♦

Proof. Fix a posreal ε . Take N large enough that

$$\text{Tail}_N(\vec{x}) \subset \text{Bal}_{\frac{\varepsilon}{2}}(\alpha) \quad \text{and} \quad \text{Tail}_N(\vec{y}) \subset \text{Bal}_{\frac{\varepsilon}{2}}(\beta).$$

Each index k has $p_k - [\alpha + \beta] = [x_k - \alpha] + [y_k - \beta]$. For each $k \geq N$, then,

$$|p_k - [\alpha + \beta]| \leq |x_k - \alpha| + |y_k - \beta| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacklozenge$$

Remark. The same thm and proof hold for addition on a normed vectorspace; simply replace $|\cdot|$ by the norm $\|\cdot\|$. □

Abbreviations. Use *WELOG* for “without essential loss of generality”, and *posint* for “positive integer”.

A sequence \vec{x} abbreviates (x_1, x_2, x_3, \dots) . Use $\text{Diam}(\vec{x})$ for the *diameter* of the set $\{x_n\}_{n=1}^{\infty}$. □

2: Mult-Cts thm. *The multiplication operation $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is continuous. Restated: Suppose $\vec{x}, \vec{y} \subset \mathbb{C}$ with $\lim(\vec{x}) = \alpha$ and $\lim(\vec{y}) = \beta$. With $p_n := x_n \cdot y_n$, then, $\lim(\vec{p}) = \alpha \cdot \beta$.* ♦

Proof. WELOG $|\beta| \leq 7$. Since \vec{x} converges, a theorem from class tells us that $\text{Diam}(\vec{x})$ is finite; WELOG

$$\dagger: \quad \forall \text{ posints } n: \quad |x_n| \leq 50.$$

For each posint n , adding and subtracting a term gives

$$\begin{aligned} x_n y_n - \alpha \beta &= x_n y_n - x_n \beta + x_n \beta - \alpha \beta \\ &= x_n [y_n - \beta] + [x_n - \alpha] \beta. \end{aligned}$$

Taking absolute-values, then upper-bounding, yields

$$\begin{aligned} \ddagger: \quad |x_n y_n - \alpha \beta| &\leq |x_n| \cdot |y_n - \beta| + |x_n - \alpha| \cdot |\beta| \\ &\leq 50 \cdot |y_n - \beta| + |x_n - \alpha| \cdot 7, \end{aligned}$$

by (\dagger) and the first sentence.

Fix a posreal ε . Since $\lim(\vec{y}) = \beta$ and $\lim(\vec{x}) = \alpha$, we can take K large enough that for each $n \in [K .. \infty)$:

$$|y_n - \beta| \leq \frac{\varepsilon/2}{50} \quad \text{and} \quad |x_n - \alpha| \leq \frac{\varepsilon/2}{7}.$$

Plugging these estimates in to (\ddagger) gives that

$$|x_n y_n - \alpha \beta| \leq 50 \cdot \frac{\varepsilon/2}{50} + \frac{\varepsilon/2}{7} \cdot 7 \stackrel{\text{note}}{=} \varepsilon,$$

for each $n \geq K$.

As this holds for every ε positive, $\lim(\vec{x} \cdot \vec{y}) \stackrel{\text{indeed}}{=} \alpha \beta$. ♦

Filename: Problems/Analysis/Calculus/add-mult-are-cts.
 latex

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