

Sets and Logic  
MHF3202 17HE

Home-A

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12Oct2021

Due **BoC, Monday, 14Feb2022**, wATMP! Write **DNE** if the object does not exist or the operation cannot be performed. NB:  $\mathbf{DNE} \neq \{\} \neq 0$ .

**Print this problem-sheet**; it forms the first two pages of your write-up, with the blanks filled in (handwritten) and the HONOR CODE signed.

**A1:** *Show no work.*

**a** The *Threeish-numbers* comprise  $\mathcal{T} := 1 + 3\mathbb{N}$ .  
 $\mathcal{T}$ -number 385  $\stackrel{\text{note}}{=} 35 \cdot 11$  is  $\mathcal{T}$ -irreducible:  $T \quad F$   
 Threeish  $N := 85$  is **not**  $\mathcal{T}$ -prime because  $\mathcal{T}$ -numbers  
 $J := \dots$  and  $K := \dots$  satisfy  
 $\lfloor \dots \rfloor \quad \lfloor \dots \rfloor$   
 that  $N \blacklozenge [J \cdot K]$ , **yet**  $N \not\blacktriangleright J$  and  $N \not\blacktriangleright K$ .  
 Also,  $\mathcal{T}$ -GCD(175, 70) =  $\dots$ .

**b** On  $\Omega := [1 .. 29] \times [1 .. 29]$ , define binary-relation  $\mathbf{C}$  by:  
 $(x, \alpha) \mathbf{C} (y, \beta)$  IFF  $x \cdot \beta \equiv_{30} y \cdot \alpha$ . Statement  
 “Relation  $\mathbf{C}$  is an **equivalence relation**” is:  $T \quad F$

Carefully TYPE your essays, double-spaced. I suggest L<sup>A</sup>T<sub>E</sub>X, but other systems are ok too.

**A2:** Let  $\mathbf{E}_n$  be the equilateral triangle with side-length  $2^n$ . This  $\mathbf{E}_n$  can be tiled in an obvious way by  $4^n$  many little-triangles [copies of  $\mathbf{E}_0$ ]; see picture on blackboard. The “punctured  $\mathbf{E}_n$ ”, written  $\widetilde{\mathbf{E}}_n$ , has its topmost copy of  $\mathbf{E}_0$  removed.

A (trape)zoid,  $\mathbf{T}$ , comprises three copies of  $\mathbf{E}_0$  glued together in a row, rightside-up, upside-down, rightside-up. [A zoid-tiling allows all three rotations of  $\mathbf{T}$ .]

**i** PROVE: For each  $n$ , board  $\widetilde{\mathbf{E}}_n$  admits a zoid-tiling.

**ii** Let  $\Delta_k$  be the equilateral triangle of sidelength  $k$ ; so  $\mathbf{E}_n$  is  $\Delta_{2^n}$ . Triangle  $\Delta_k$  comprises  $k^2$  little-triangles. For what values of  $k$  does  $\Delta_k$  admit a zoid-tiling? For which  $k$  does  $\widetilde{\Delta}_k$  admit a zoid-tiling?

**iii** An *Lmino* (pron. “ell-mino”) comprises three  $\blacksquare$  squares in an “L” shape (all four orientations are allowed). Let  $\mathbf{S}_n$  be the  $2^n \times 2^n$  square board, comprising  $4^n$  *squares* (little squares). Have  $\widetilde{\mathbf{S}}_n$  be the board with one

corner square removed. Shown in class is an inductive proof that each  $\widetilde{\mathbf{S}}_n$  is Lmino-tilable (by  $[4^n - 1]/3$  Lminos, of course). Further, with  $\mathbf{S}'_n$  denoting  $\mathbf{S}_n$  with an arbitrary puncture, we proved that every  $\mathbf{S}'_n$  is Lmino-tilable.

Generalize this to three-dimensions. Let  $\mathbf{C}_n$  denote the  $2^n \times 2^n \times 2^n$  cube,  $\widetilde{\mathbf{C}}_n$  the corner-punctured cube, and let  $\mathbf{C}'_n$  be  $\mathbf{C}_n$  but with an arbitrary *cubie* removed.

What is the 3-dimensional analog of an Lmino? Calling it a “3-mino”, how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: Every  $\mathbf{C}'_n$  admits a 3-mino-tiling. [Provide also pictures showing your ideas.]

**iv** Generalize to  $K$ -dim(ensional) space, with  $\mathbf{C}_{n,K}$  being the  $2^n \times \dots \times 2^n$  cube, having  $[2^n]^K = 2^{nK}$  many  $K$ -dim'al cubies. As before, let  $\mathbf{C}'_{n,K}$  be  $\mathbf{C}_{n,K}$  with an arbitrary cubie removed.

What is your  $K$ -mino with which you will tile, and how many cubies does it have? (So a 2-mino is our Lmino.) PROVE: Every  $\mathbf{C}'_{n,K}$  admits a  $K$ -mino-tiling.

**A3:** Recall *Rabbits and Lights* from the zoomester’s beginning: To your right are lights  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots$ . Each light has a toggle button; Press&release: the light illuminates; P&R again, it is extinguished.

Off to your left is a queue of rabbits; so we have

$$\dots \mathcal{R}_3 \mathcal{R}_2 \mathcal{R}_1 \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4, \dots$$

All the lights are initially off. If rabbit- $\alpha$  (i.e.  $\mathcal{R}_\alpha$ ) jumps, then he will hop on lights  $\mathcal{L}_\alpha, \mathcal{L}_{2\alpha}, \mathcal{L}_{3\alpha}, \dots$ , turning them all on. If rabbit- $\beta$  now jumps, he will change the state of lights  $\beta, 2\beta, 3\beta, \dots$ , turning some on, and some off.

*A Map f.* A (finite or infinite) set  $R = \{\alpha_1, \alpha_2, \dots\}$  of rabbit-indices is an element of powerset  $\mathcal{P} := \mathcal{P}(\mathbb{Z}_+)$ . After those rabbits jump, we have a (finite or infinite) set  $L = \{\beta_1, \beta_2, \beta_3, \dots\}$  of indices of illuminated lights. Define  $f: \mathcal{P} \rightarrow \mathcal{P}$  by  $f(R) := L$ .

Our first-day class showed [involvement argument, and re-argued using the divisor-count  $\tau$ -fnc] that  $f(\mathbb{Z}_+)$  is the set  $\{1, 4, 9, \dots\}$  of squares. Evidently  $f(\emptyset) = \emptyset$  and  $f(\{1, 2\}) = \text{Odds}$ . □

**Q1** For each of the following questions, produce *either* a **CEX** [counterexample] *or* a **formal proof**.  
 Is  $f$  injective? Is  $f$  surjective?

**Q2** For  $L \in \text{Range}(f)$ , give an algorithm to produce an  $R$  for which  $f(R) = L$ . If you program, can you implement your algorithm in computer code?

**Q3** Produce a (*non-trivial*) commutative, associative binop  $\$: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$  which satisfies

$$\forall R, R': f(R \$ R') = f(R) \$ f(R').$$

What can you tell me about this binary operator?

**Q4** What is the  $f$ -fixed-point set; those  $R$  with  $f(R) = R$ ?

What can you say about the dynamics of  $f$ ? —does it have periodic points of order 2? 3? ...?

What is  $f(f(\mathbb{Z}_+)) \stackrel{\text{note}}{=} f(\text{Squares})$ ? (Conjecture? Computer simulation?)

**A4:** Produce an *interesting* Induction or PHP problem, then solve it (hopefully, elegantly).

**A1:** \_\_\_\_\_ 40pts

**A2:** \_\_\_\_\_ 125pts

**A3:** \_\_\_\_\_ 110pts

**A4:** \_\_\_\_\_ 20pts

**Total:** \_\_\_\_\_ 295pts

**HONOR CODE:** “I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague).” *Name/Signature/Ord*

\_\_\_\_\_ Ord:  
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