Staple!

**Team: A ___**

Sets and Logic MHF3202 3E07 **Home-A** Prof. JLF King Wednesday 05Feb2020

Due **BoC, Monday, 10Feb2020**, wATMP! Please *fill-in* every *blank* on this sheet. Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$.

**A1:** *Show no work. Simply fill-in each blank on the problem-sheet.*

___ **a** Given sets with cardinalities $|B| = 8$ and $|E| = 5$, the number of non-constant fncs in $B^E$ is ______________.

___ **b** Using *only* symbols $\boldsymbol{H}$, $\boldsymbol{D}$, $\wedge$, $\vee$, $\neg$, $T$, $F$, ], [, rewrite (in simplest form) expression $[[H \Rightarrow D] \Rightarrow H]$ as ____________________________. Ditto, rewrite $[H \Rightarrow [D \Rightarrow H]]$ as ____________________.

___ **c** *$\forall x,z \in \mathbb{Z}$ with $x < z$, $\exists y \in \mathbb{Z}$ st.: $x < y < z$.* $T$ $F$

*$\forall x,z \in \mathbb{Q}$ with $x \neq z$, $\exists y \in \mathbb{R}$ st.: $x < y < z$.* $T$ $F$

*For all sets $\Omega$, there exists a fnc $f: \mathbb{R} \to \Omega$.* $T$ $F$

___ **d** In $[5x^2 + 4y + z^3 + 7]^{20}$, compute these coeffs:

$\text{Coeff}(x^6 z^8) =$ ____________________________.

$\text{Coeff}(y^5 z^6) =$ ____________________________

[You may write answers as a product numbers, powers and multinomial-coeffs.]

___ **e** The number of ways of picking 42 objects from 70 types is $\left[\!\left[\begin{matrix}70\\42\end{matrix}\right]\!\right] \stackrel{\text{Binom}}{=\!=}_{\text{coeff}} \binom{\quad\quad}{\quad\quad}$. And $\left[\!\left[\begin{smallmatrix}70\\42\end{smallmatrix}\right]\!\right] = \left[\!\left[\begin{smallmatrix}T\\N\end{smallmatrix}\right]\!\right]$, where $T =$ ______ $\neq 70$, and $N =$ ______.

*For the two essay questions, carefully* TYPE, *double spaced, grammatical solns. I suggest* LaTeX, *but other systems are ok too.*

**A2:** Define a sequence $\vec{\mathbf{b}} = (b_0, b_1, b_2, \ldots)$ by $b_0 := 0$ and $b_1 := 3$ and

$$\dagger: \quad b_{n+2} := 7b_{n+1} - 10b_n, \quad \text{for } n = 0, 1, \ldots.$$

Use induction to prove, for each natnum $k$, that

$$\ddagger: \quad b_k = 5^k - 2^k.$$

***Further:*** Given recurrence (†) and initial conditions, *explain* how you could have discovered/computed the numbers 5 and 2 in the (‡) formula.

Can you generalize to getting a (‡)-like formula when the recurrence is $\boxed{b_{n+2} := \mathbf{S}b_{n+1} - \mathbf{P}b_n}$, for arbitrary real-number coefficients $\mathbf{S}$ and $\mathbf{P}$?

**A3:** Let $\mathbf{E}_n$ be the equilateral triangle with side-length $2^n$. This $\mathbf{E}_n$ can be tiled in an obvious way by $4^n$ many little-triangles [copies of $\mathbf{E}_0$]; see picture on blackboard. The **"*punctured* $\mathbf{E}_n$"**, written $\widetilde{\mathbf{E}_n}$, has its topmost copy of $\mathbf{E}_0$ removed.

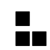
A ***(trape)zoid***, $\mathbf{T}$, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up. [A ***zoid-tiling*** allows all three rotations of $\mathbf{T}$.]

**i** PROVE: *For each $n$, board $\widetilde{\mathbf{E}_n}$ admits a zoid-tiling.*

**ii** Let $\Delta_k$ be the equilateral triangle of sidelength $k$; so $\mathbf{E}_n$ is $\Delta_{2^n}$. Triangle $\Delta_k$ comprises $k^2$ little-triangles.

*For what values of $k$ does $\Delta_k$ admit a zoid-tiling?*

*For which $k$ does $\widetilde{\Delta_k}$ admit a zoid-tiling?*

**iii** An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n \times 2^n$ square board, comprising $4^n$ ***squaries*** (little squares). Have $\widetilde{\mathbf{S}_n}$ be the board with one corner squarie removed. Shown in class is an inductive proof that each $\widetilde{\mathbf{S}_n}$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, V. proves that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n \times 2^n \times 2^n$ cube, $\widetilde{\mathbf{C}_n}$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

*What is the 3-dimensional analog of an Lmino?* Calling it a **"*3-mino*"**, how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

**iv** Generalize to $K$-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n \times \overset{K}{\ldots} \times 2^n$ cube, having $[2^n]^K = 2^{nK}$ many $K$-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your $K$-**mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a $K$-mino–tiling.*

End of Home-A

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| **A1:** | ___ ___ ___ | 135pts |
| **A2:** | ___ ___ | 60pts |
| **A3:** | ___ ___ ___ | 130pts |
| **Total:** | ___ ___ ___ | 325pts |

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

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