Sets and Logic MHF3202 2787 | **Home-A** | Prof. JLF King Thursday, 20Sep2018

Due **BoC, Monday, 24Sep2018**, wATMP! Please *fill-in* every *blank* on this sheet. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

A1: *Show no work. Simply fill-in each blank on the problem-sheet.*

a [With $\mathcal{P}()$ the *powerset* operator, let $S :=$ 3-stooges.] Then $|\mathcal{P}(S)| = 2^3 = 8$ and $|\mathcal{P}(\mathcal{P}(S))| = 2^{2^3} = 2^8 = 256$.

b Using *only* symbols $H, D, \wedge, \vee, \neg, T, F,], [$, rewrite (in simplest form) expression $[[H \Rightarrow D] \Rightarrow H]$ as H. Ditto, rewrite $[H \Rightarrow [D \Rightarrow H]]$ as T.

c $\forall x,z \in \mathbb{Z}$ *with* $x < z$, $\exists y \in \mathbb{Z}$ *st.:* $x < y < z$. T (F)

$\forall x,z \in \mathbb{Q}$ *with* $x \neq z$, $\exists y \in \mathbb{R}$ *st.:* $x < y < z$. T (F)

For all sets Ω, *there exists a fnc* $f\colon\mathbb{R}\to\Omega$. T (F)

d The coeff of $x^7 y^{12}$ in $[5x + y^3 + 1]^{30}$ is $5^7 \cdot \binom{30}{7,\,4,\,19}$, *since* $\frac{12}{3}$ *is* 4, and...

[You may write in form number times multinomial-coeff. You can leave the multinomial-coeff as such, or write ITOf factorials.]

...we are extracting the $[5x]^7 \cdot [y^3]^4 \cdot 1^{30-[7+4]}$ terms.

e The number of ways of picking 42 objects from 70 types is $\left[\!\left[\begin{matrix}70\\42\end{matrix}\right]\!\right] \overset{\text{Binom}}{\underset{\text{coeff}}{=}} \Big(\quad \Big)$. And $\left[\!\left[\begin{matrix}70\\42\end{matrix}\right]\!\right] = \left[\!\left[\begin{matrix}T\\N\end{matrix}\right]\!\right]$, where $T=$ ______ $\neq 70$, and $N=$ ______.

Soln: We use $70 - 1 = 69$ separators. So

$$\left[\!\left[\begin{matrix}70\\42\end{matrix}\right]\!\right] = \binom{69+42}{69,\ 42} = \binom{111}{69,\ 42}.$$

This also equals

$$\binom{111}{42\,,69} = \binom{[43-1]+69}{43-1,\ 69},$$

so $T = 43$ and $N = 69$.

For the two essay questions, carefully TYPE, *double-or-triple–spaced, grammatical solns. I suggest* LaTeX, *but other systems are ok too.*

A2: **i** On a 7×7 chessboard, 22 rooks are placed. Prove there exists a ***friendly*** 4-set of rooks. [I.e, on 4 distinct rows and on 4 distinct columns. Shorthand: You may use ***clump*** for "friendly 4-set".] Illustrate the concepts in your proof with large, useful Pictures. [*Hint:* PHP]

ii On a 7×7 chessboard, 23 rooks are placed. Prove: ***Either*** *there exists a friendly* 5*-set, or a disjoint-pair of friendly* 4*-sets.* [An n-set of rooks is ***friendly*** if the rooks lie on n distinct rows, and n distinct columns. Shorthand: You may use ***double-clump*** for "*disjoint-pair of friendly* 4*-sets*".]

Soln: See the "Rooks" problem in the "Problem List for SeLo (pdf)" link of our SeLo course page.

STRONGER: Prove there *always* exists a double-clump, unconditionally.

A3: Define a sequence $\vec{\mathbf{b}} = (b_0, b_1, b_2, \ldots)$ by $b_0 := 0$ and $b_1 := 3$ and

†: $\quad b_{n+2} := 7b_{n+1} - 10b_n$, for $n = 0, 1, \ldots$.

Use induction to prove, for each natnum k, that

‡: $$b_k = 5^k - 2^k.$$

Further: Given recurrence (†) and initial conditions, *explain* how you could have discovered/computed the numbers 5 and 2 in the (‡) formula.

Can you generalize to getting a (‡)-like formula when the recurrence is $\boxed{b_{n+2} := \mathbf{S}b_{n+1} - \mathbf{P}b_n}$, for arbitrary real-number coefficients $\mathbf{S}$ and $\mathbf{P}$?

Soln: 2-term lin-recurr. Let $D_k := 5^k - 2^k$. Then $b_0 \overset{\text{def}}{=} 0 = 1 - 1 \overset{\text{note}}{=} D_0$. Then $b_1 \overset{\text{def}}{=} 3 = 5 - 2 \overset{\text{note}}{=} D_1$; so the base cases hold..

Fix natnum n so that ($‡_n$) and ($‡_{n+1}$) hold. Then

$$\begin{aligned} b_{n+2} &\overset{\text{def}}{=} 7b_{n+1} - 10b_n \\ &\overset{\text{Ind.hyp}}{=} 7\cdot[5^{n+1} - 2^{n+1}] - [5\cdot 2][5^n - 2^n] \\ &= [7-2]\cdot 5^{n+1} - [7-5]\cdot 2^{n+1} \\ &= 5\cdot 5^{n+1} - 2\cdot 2^{n+1} \overset{\text{note}}{=} D_{n+2}. \quad \blacklozenge \end{aligned}$$

Whence 5 and 2? Observe that $b_{n+2} - 7b_{n+1} + 10b_n = 0$ is the recurrence rewritten. Corresponding polynomial $f(x) := x^2 - 7x + 10$ factors as $f(x) = [x-5][x-2]$. Etc.

Addendum. This is solved more generally at the "*Linear Recurrence using matrices*" link on our http://squash.1gainesville.com/teaching.html page.

End of Home-A

A1:	___ ___ ___	140pts
A2:	___ ___	90pts
A3:	___ ___	60pts
Total:	___ ___ ___	290pts