Staple!

Team A____

Probability II MAP6473 4810 | **Home-A** | Prof. JLF King *Touch*: 4Aug2016

Please. General instructions/notation are on the CHECKLIST and "Conditional probability & conditional measures". Note: MG means Martingale, and ST means Stopping Time.

A1: Consider a MG $\vec{\mathsf{Y}}$ with pointwise bounds

1: $$|\mathsf{Y}_0| \leqslant 7 \quad\text{and}\quad \forall n:\ |\mathsf{Y}_{n+1} - \mathsf{Y}_n| \leqslant 7\,.$$

Suppose that β is a ST with $\mathsf{E}(\beta) < \infty$.

Prove that Y_β is integrable and $\mathsf{E}(\mathsf{Y}_\beta) = \mathsf{E}(\mathsf{Y}_0)$.

A2: Consider an independent random-walk on the integers, where each step-probability depends on both position and time.

A ***3-spread*** $D()$ is a mean-zero random variable with support on $J := [-3\,..\,{+3}]$. That is,

2: $$\sum_{j\in J} \mathsf{P}(D{=}j) \;=\; 1 \quad\text{and}$$

3: $$\mathsf{E}(D) \stackrel{\text{note}}{=\!=} \sum_{j\in J} j\cdot\mathsf{P}(D{=}j) \;=\; 0\,.$$

For each *time* $n \in \mathbb{Z}_+$ and each *position* $p \in \mathbb{Z}$, suppose that we have a 3-spread $D_{n,p}$, and all these random variables are mutually independent. Then our random-walk is $\vec{\mathsf{S}}$, where $\mathsf{S}_0 \equiv 0$ (we start at the origin) and

$$\mathsf{S}_{n+1} \;:=\; \mathsf{S}_n + D_{n+1,\,\mathsf{S}_n}\,.$$

Let $\tau()$ be the stopping time where the random-walk first hits position "5". Use the preceding problem to prove that $\mathsf{E}(\tau)$ must be infinite. [*Hint:* For a MG, be precise with your definition of $\mathcal{G}_n$.]

A3: Let $\vec{\mathsf{X}} := (\mathsf{X}_n)_{n=1}^{\infty}$ be a fair-coin process. Given a finite (but not necessarily bounded) stopping-time $\tau()$, we get a r.var X_τ and, more generally, a "random vector" $\mathsf{Y} := (\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_\tau)$.

Fix a particular number $z \in [0,1]$; for specificity, consider $z := 1/\boldsymbol{\pi}$. Cook up a τ so that, from Y, you can compute a *Yes!* or *No!*, so that the probability you say *Yes!* is exactly z. [*Hint:* Express z in binary as $z = \sum_{n=1}^{\infty} b_n/2^n$.]

a What is the expected running-time of your algorithm? –can you make it finite?

b Now the coin is a $p+q=1$ *biased*-coin and **you don't know the bias!** (You do know that $p,q>0$.) Cook up an experiment, as above, which answers *Yes!* with prob$=z$. (Note that, since $\tau()$ is a.e-finite, you won't know what the bias of the coin was, even though you shouted an answer.)

In terms of p and q, what is the expected running-time of your algorithm?

A4: Suppose $(\vec{\mathsf{S}}, \vec{\mathcal{G}})$ is a subMG. Produce a MG $\vec{\mathsf{Y}}$ adapted to $\vec{\mathcal{G}}$, and positive process $\vec{P}$ so that

d1: $\mathsf{S}_n \;=\; \mathsf{Y}_n + P_n \quad$ (for $n = 0,1,2,\ldots$).

d2: $0 = P_0 \leqslant P_1 \leqslant P_2 \leqslant P_3 \leqslant \ldots$

d3: Each P_j is measurable w.r.t $\mathcal{G}_{j-1}$.

[*Hint:* What does conditioning $\mathsf{S}_n - \mathsf{S}_{n-1}$ on $\mathcal{G}_{n-1}$ say about $\vec{P}$? Now try Astronomy...]

c Suppose that $\mathsf{S}_n = \widehat{\mathsf{Y}}_n + \widehat{P}_n$ is *another* decomposition fulfilling (d1,d2,d3). Prove that

$$\widehat{\mathsf{Y}}_n \stackrel{\text{a.e}}{=} \mathsf{Y}_n \quad\text{and}\quad \widehat{P}_n \stackrel{\text{a.e}}{=} P_n\,,$$

for all n. The decomposition is *unique*.

End of Home-A

A1:	___ ___	70pts
A2:	___ ___	70pts
A3:	___ ___	70pts
A4:	___ ___	70pts
Total:	___ ___ ___	280pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

Ord: ....

Ord: ....