

**A1:** *Show no work.*

**a** Cubic polynomial  $h(x) := [x + 5][x - 11][x + 37]$  has  $K$  many roots in  $\mathbb{Z}_8$ , and  $N$  many roots in  $\mathbb{Z}_{120}$ , where  $K = \dots$  and  $N = \dots$ . [Hint: CRT.]

**Root Soln:** Note  $120 = 8 \cdot 3 \cdot 5$ . Recall

$$\begin{aligned} h(x) &= [x + 5] \cdot [x - 11] \cdot [x + 37]. \quad \text{Thus,} \\ h(x) &\equiv_8 [x - 3] \cdot [x - 3] \cdot [x - 3] \quad \text{and} \\ h(x) &\equiv_3 [x - 1] \cdot [x - 2] \cdot [x - 2] \quad \text{and} \\ h(x) &\equiv_5 [x - 0] \cdot [x - 1] \cdot [x - 3]. \end{aligned}$$

Letting  $\#_M$  denote the number of  $h$ -roots in  $\mathbb{Z}_M$ , we have  $\#_3 = 2$  and  $\#_5 = 3$ ; for when  $p$  is prime then  $\mathbb{Z}_p$  is a field, whence  $\mathbb{Z}_p[x]$  has unique factorization.

In  $\mathbb{Z}_8$ , zero has four cube-roots, namely  $0, \pm 2, 4$ , whence  $h$  has four roots in  $\mathbb{Z}_8$ ; i.e.,  $\#_8 = 4$ . Thus in  $\mathbb{Z}_{120}$ , our  $h$  has  $\#_8 \cdot \#_3 \cdot \#_5 = 4 \cdot 2 \cdot 3 = 24$  roots, courtesy CRT.

**b** Euler  $\varphi(36300) = 2^A \cdot 3^B \cdot 5^C \cdot 7^D \cdot 11^E$ , where  $A = \dots$ ,  $B = \dots$ ,  $C = \dots$ ,  $D = \dots$ ,  $E = \dots$ .

**$\varphi$ -Soln:** Evidently  $36300 = 3 \cdot 121 \cdot 10^2 = 2^2 \cdot 3 \cdot 5^2 \cdot 11^2$ . So

$$\begin{aligned} \varphi(36300) &= 2[2 - 1] \cdot [3 - 1] \cdot 5[5 - 1] \cdot 11[11 - 1] \\ &= 2 \cdot 2 \cdot 5 \cdot 4 \cdot 11 \cdot 10 \\ &= 2^5 \cdot 3^0 \cdot 5^2 \cdot 7^0 \cdot 11^1. \end{aligned}$$

Thus  $(A, B, C, D, E) = (5, 0, 2, 0, 1) = \text{Nevada}$ .

**c** Fix a prime  $q$  and natnums  $J$  and  $R$ . Then a closed-formula

for  $\sigma_J$  is:  $\sigma_J(q^R) = \dots$

**GS Soln:** With  $r := q^J$ , we sum finite geometric series:

$$\begin{aligned} \sigma_J(q^R) &= 1 + r + r^2 + \dots + r^R \\ &= \frac{r^{R+1} - 1}{r - 1} = \frac{q^{J(R+1)} - 1}{q^J - 1}. \end{aligned}$$

Apply the [correct] CF; leave your

answer as a product:  $\sigma_2(140) = \dots$

**MF Soln:** Note  $140 = 2^2 \cdot 5^1 \cdot 7^1$ . Thus  $\sigma_2(140)$  equals

$$\frac{2^6 - 1}{2^2 - 1} \cdot \frac{5^4 - 1}{5^2 - 1} \cdot \frac{7^4 - 1}{7^2 - 1} = 21 \cdot 26 \cdot 50 = 27300.$$

Note: WolframAlpha uses `DivisorSigma(2, 140)`

**d** The *Chris-numbers* comprise  $\mathcal{C} := 1 + 3\mathbb{N}$ .

$\mathcal{C}$ -number  $385 \stackrel{\text{note}}{=} 35 \cdot 11$  is  $\mathcal{C}$ -irreducible:  $T \quad F$ .

**Irr Soln:** *False.* Note  $35 = 7 \cdot 5$ . So  $385 = 7 \cdot [5 \cdot 11]$  is a non-trivial *Chris*-factorization of 385.

$\mathcal{C}$ -number  $N := 85$  is **not**  $\mathcal{C}$ -prime because  $\mathcal{C}$ -numbers  $J := \dots$  and  $K := \dots$  satisfy that  $N \blacklozenge [J \cdot K]$ , **yet**  $N \not\blacklozenge J$  and  $N \not\blacklozenge K$ .

**Prime Solution:** An integer  $k$  is **3Neg** if  $k \equiv_3 -1$  and **3Pos** if  $k \equiv_3 +1$ . Note  $85 = 5 \cdot 17$  is a product of two 3Neg primes. We simply need to place one prime in  $J$  and the other in  $K$ . Hence a soln is  $(J, K) := (5 \cdot 5, 17 \cdot 17)$ .

A more general soln is  $(J, K) := (5p, 17q)$ , where  $p, q$  are 3Neg numbers st.  $p \not\blacklozenge 17$  and  $q \not\blacklozenge 5$ . Letting  $p = q := 2$  yields  $(J, K) := (10, 34)$  as a smallest soln.

**e** Multinomial  $\binom{9}{4, 2, 3} = \dots = \dots$

[Note: Write your ans. ITOF factorials, then **also** write it as a single integer, or product of two, **without** factorials.]

**Nomial Soln:** Directly,  $\binom{9}{4, 2, 3} = \frac{9!}{4! \cdot 2! \cdot 3!}$ . Computing,  $\binom{9}{4, 2, 3} = \binom{9}{4} \cdot \binom{5}{2} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} = 126 \cdot 10$ .  
(multinom-coeff 9 '(4 2 3)) => 1260

OYOP: In grammatical *English sentences*, write your essays on every **third** line (usually), so that I can easily write between the lines. Do **not** restate the question. Start each essay on a **new** sheet of paper.

**A2:** State Wilson's Thm. Carefully prove Wilson's Thm.

*More on next page...*

**A3:** Let  $T_d := 18^d + 1$  for  $d = 3, 5, 7, 9, 11, \dots$ . Prove that each such  $T_d$  is composite. **NB:**  $T_3 = 5833 = [19 \cdot 307]$ .  
 [Hint: Look at  $T_{\text{Odd}} \pmod{N}$ , for an appropriate  $N$ .]

**Soln 1:** If  $N$  divides  $T_{\text{Odd}}$ , then it likely divides  $T_1$  too. Our  $T_1 = 19$ , and note that  $T_{\text{Odd}} \equiv_{19} [-1]^{\text{Odd}} + 1 = 0$ .

Finally,  $n \mapsto T_n$  is strictly increasing, so for  $d > 1$ , our  $T_d > T_1 = 19$ , hence 19 is a **proper** divisor of  $T_d$ .

**Soln 2:** If  $N$  divides  $T_{\text{Odd}}$ , then it divides differences  $T_{d+2} - T_d$ . Note  $T_{n+2} = 18^2[T_n - 1] + 1$ . Thus

$$\begin{aligned} T_{n+2} - T_n &= [18^2 - 1] \cdot [T_n - 1] \\ &= 19 \cdot 17 \cdot [T_n - 1]. \end{aligned}$$

Finally,  $T_1 \bullet 19$ , so all  $T_{\text{Odd}} \bullet 19$ , etc..

**Soln 3:** For  $n \geq 1$ , note  $[A - B]$  divides  $[A^n - B^n]$  since  $[A^n - B^n]$  equals product

$$[A - B] \cdot [A^{n-1}B^0 + A^{n-2}B^1 + \dots + A^0B^{n-1}].$$

With  $d$  odd and  $B = -1$ , this says that

$$[A + 1] \text{ divides } [A^d + 1].$$

Plugging in  $A := 18$  says that  $19 \bullet T_d$ , etc..

End of Class-A

**A1:**    \_\_\_ \_\_\_ \_\_\_    125pts

**A2:**        \_\_\_ \_\_\_    45pts

**A3:**        \_\_\_ \_\_\_    35pts

**Total:**    \_\_\_ \_\_\_ \_\_\_    205pts