Combinatorics MAD4203 3214 | **Class-A** | Prof. JLF King 29Sep2017

A3: Short answer. Show no work.
Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

a Prof. King believes that writing in complete, coherent sentences is crucial in communicating Mathematics, improves posture, and whitens teeth. Circle one:

True! **Yes!** ***wH'at S a?sEnTENcE***

b $B^E = \sum_{j=0}^{59} \binom{59}{j} \cdot 4^{2j}$, for posints $B=$ ____ & $E=$ ____.

Soln: Note $4^{2j} = 16^j \cdot 1^{59-j}$. By the Binomial theorem, the sum equals $[16+1]^{59}$, so $B = 17$ and $E = 59$.

c The physics lab has atomic *zinc*, *tin*, *silver* and *gold*. I'm allowed to take 6 atoms, so I have [expressed as single integer] ____ many possibilities.

Types: Computing, $\left[\!\!\binom{6}{4}\!\!\right] = \binom{6+4-1}{6,\ 4-1} = \binom{9}{6,\ 3} = 84$.

d There are $\binom{K}{J}$ many [diagonal] lattice-paths from point $(0,2)$ to $(21,7)$, where $K=$ ____ and $J=$ ____.

Such a path is ***bad***, if it touches the x-axis. And $|\text{BAD}| = \binom{N}{L}$, where $N=$ ____ and $L=$ ____.

Paths soln. Use U for an up-diag and D for a down-diag [adding $(1,+1)$, respectively, $(1,-1)$ to the current location]. The total number of Us and Ds is 21, so $K = N = 21$.

Going from $\mathbf{A} := (0,2)$ to $\mathbf{B} := (21,7)$, needs $7-2=5$ more Us than Ds. Thus

$$[5 + \#\mathsf{D}] + \#\mathsf{D} \;=\; 21\,,$$

so $\#\mathsf{D} = \frac{21-5}{2} = 8$. We can take J to be 8 or 8+5 = 13.

A bad path first touches the x-axis at some point. Reflecting the rest of the path across the x-axis has the path terminate at $\mathbf{B}' := (21,-7)$ rather than at $\mathbf{B}$. This path needs $2-(-7) = 9$ more Ds than Us. Thus $\#\mathsf{U}+[9+\#\mathsf{U}] = 21$, so $\#\mathsf{U} = 6$. Take L to be 6 or 6+9 = 15.

Aside: So the number of good paths is $\binom{21}{8} - \binom{21}{6}$.

OYOP: *In grammatical English* ***sentences****, write your essays on every* ***third*** *line* (usually)*, so that I can easily write between the lines.* Start each essay on a new sheet-of-paper. Please number the pages "1 of 57", "2 of 57"... (or "1/57", "2/57"...) I suggest you put your name on each sheet.

A4: Give a careful bijective proof of:

Thm: *Fix a natnum $N \geqslant 3$. Then*

$$*: \qquad [\![N!3]\!] \cdot 2^{N-3} \;=\; \sum_{k=3}^{N} [\![k!3]\!] \cdot \binom{N}{k}\,.$$

Soln: Each side counts the number of committees comprising a Pres., VP and Secretary (and possibly no other members) from set of N people.

LhS$(*)$ picks Pres. then VP, then Secr., $[\![N!3]\!]$ triples; *then* it picks the rest of the committee, which has 2^{N-3} possible fill-outs.

The RhS$(*)$ first picks the committee of $k \geqslant 3$ members. Then from these k folks, it picks an ordered-triple **(***Pres., VP, Secr.***)**.

A5: Prove, for each natnum N, that

$$\sum_{k=0}^{N} \left[\binom{N}{k}^2 \right] \;=\; \binom{2N}{N}\,.$$

Binom id: In choosing N objects from $[1\,..\,2N]$, let k denote the number we picked in $[1\,..\,N]$, the first half. Then we must *avoid* taking some k objects from the second half. The number of ways of doing this $\binom{N}{k} \cdot \binom{N}{k}$.

End of Class-A

A3:	___ ___	70pts
A4:	___ ___	35pts
A5:	___ ___	35pts
Total:	___ ___ ___	140pts